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Term Structure of Interest Rates

TERM STRUCTURE MODELS are particularly simple, since bond prices are just the expected value of the discount factor. In equations, the price at time t of a zero-coupon bond that comes due at time $t+j$ is $P_t^{(j)} = E_t(m_{t,\,t+j})$. Thus, once you specify a time-series process for a one-period discount factor $m_{t,\,t+1}$, you can in principle find the price of any bond by chaining together the discount factors and finding $P_t^{(j)} = E_t(m_{t,\,t+1}m_{t+1,\,t+2}\ldots m_{t+j-1,\,t+j})$. As with option pricing models, this chaining together can be hard to do, and much of the analytical machinery in term structure models centers on this technical question. As with option pricing models, there are two equivalent ways to do the chaining together: Solve the discount factor forward and take an integral, or find a partial differential equation for prices and solve it backwards from the maturity date.

19.1 Definitions and Notation

A quick introduction to bonds, yields, holding period returns, forward rates, and swaps.

$p_t^{(N)}$ = log price of N period zero-coupon bond at time t.

$y^{(N)} = -\frac{1}{N}p^{(N)}$ = log yield.

$\text{hpr}_{t+1}^{(N)} = p_{t+1}^{(N-1)} - p_t^{(N)}$ = log holding period return.

$\text{hpr} = \frac{dP(N,t)}{P} - \frac{1}{P}\frac{\partial P(N,t)}{\partial N}\,dt$ = instantaneous return.

$f_t^{(N\to N+1)} = p_t^{(N)} - p_t^{(N+1)}$ = forward rate.

$f(N,t) = -\frac{1}{P}\frac{\partial P(N,t)}{\partial N}$ = instantaneous forward rate.

Bonds

The simplest fixed-income instrument is a *zero-coupon bond.* A zero-coupon bond is a promise to pay one dollar (a nominal bond) or one unit of the consumption good (a real bond) on a specified date. I use a superscript in parentheses to denote maturity: $P_t^{(3)}$ is the price of a three-year zero-coupon bond. I will suppress the t subscript when it is not necessary.

I denote logs by lowercase symbols, $p_t^{(N)} = \ln P_t^{(N)}$. The log price has a nice interpretation. If the price of a one-year zero-coupon bond is 0.95, i.e., 95¢ per dollar face value, the log price is $\ln(0.95) = -0.051$. This means that the bond sells at a 5% discount. Logs also give the continuously compounded rate. If we write $e^{rN} = 1/P^{(N)}$, then the continuously compounded rate is $rN = -\ln P^{(N)}$.

Coupon bonds are common in practice. For example, a $100 face value 10-year coupon bond may pay $5 every year for 10 years and $100 at 10 years. (Coupon bonds are often issued with semiannual or more frequent payments, $2.50 every six months for example.) We price coupon bonds by considering them as a portfolio of zeros.

Yield

The yield of a bond is the fictional, constant, known, annual, interest rate that justifies the quoted price of a bond, assuming that the bond does not default. It is not the rate of return of the bond. From this definition, the yield of a zero-coupon bond is the number $Y^{(N)}$ that satisfies

$$P^{(N)} = \frac{1}{\left[Y^{(N)}\right]^N}.$$

Hence

$$Y^{(N)} = \frac{1}{\left[P^{(N)}\right]^{1/N}}, \qquad y^{(N)} = -\frac{1}{N}p^{(N)}.$$

The latter expression nicely connects yields and prices. If the price of a 4-year bond is -0.20 or a 20% discount, that is 5% discount per year, or a yield of 5%. The yield of any stream of cash flows is the number Y that satisfies

$$P = \sum_{j=1}^{N} \frac{CF_j}{Y^j}.$$

In general, you have to search for the value Y that solves this equation, given the cash flows and the price. So long as all cash flows are positive, this is fairly easy to do.

As you can see, *the yield is just a convenient way to quote the price.* In using yields we make *no* assumptions. We do not assume that actual interest rates are known or constant; we do not assume the actual bond is default-free. Bonds that may default trade at lower prices or higher yields than bonds that are less likely to default. This only means a higher return if the bond happens not to default.

Holding Period Returns

If you buy an N-period bond and then sell it—it has now become an $(N-1)$-period bond—you achieve a return of

$$HPR_{t+1}^{(N)} = \frac{\$\text{back}}{\$\text{paid}} = \frac{P_{t+1}^{(N-1)}}{P_t^{(N)}} \tag{19.1}$$

or, of course,

$$\text{hpr}_{t+1}^{(N)} = p_{t+1}^{(N-1)} - p_t^{(N)}.$$

We date this return (from t to $t+1$) as $t+1$ because that is when you find out its value. If this is confusing, take the time to write returns as $HPR_{t\to t+1}$ and then you will never get lost.

In continuous time, we can easily find the instantaneous holding period return of bonds with fixed maturity *date* $P(T,t)$

$$\text{hpr} = \frac{P(T,t+\Delta) - P(T,t)}{P(T,t)},$$

and, taking the limit,

$$\text{hpr} = \frac{dP(T,t)}{P}.$$

However, it is nicer to look for a bond pricing function $P(N,t)$ that fixes the *maturity* rather than the *date*. As in (19.1), we then have to account for the fact that you sell bonds that have shorter maturity than you buy:

$$\begin{aligned}\text{hpr} &= \frac{P(N-\Delta,t+\Delta) - P(N,t)}{P(N,t)} \\ &= \frac{P(N-\Delta,t+\Delta) - P(N,t+\Delta) + P(N,t+\Delta) - P(N,t)}{P(N,t)},\end{aligned}$$

and, taking the limit

$$\text{hpr} = \frac{dP(N,t)}{P} - \frac{1}{P}\frac{\partial P(N,t)}{\partial N}\,dt. \tag{19.2}$$

Forward Rate

The forward rate is defined as the rate at which you can contract *today* to borrow or lend money starting at period N, to be paid back at period $N+1$.

You can synthesize a forward contract from a spectrum of zero-coupon bonds, so the forward rate can be derived from the prices of zero-coupon bonds. Here is how. Suppose you buy one N-period zero-coupon bond and simultaneously sell $x(N+1)$-period zero-coupon bonds. Let us track your cash flow at every date:

	Buy N-period zero	Sell x $(N+1)$-period zeros	Net cash flow
Today 0:	$-P^{(N)}$	$+\,x\,P^{(N+1)}$	$x\,P^{(N+1)} - P^{(N)}$
Time N:	1		1
Time $N+1$:		$-x$	$-x$

Now, choose x so that today's cash flow is zero:

$$x = \frac{P^{(N)}}{P^{(N+1)}}.$$

You pay or get nothing today, you get \$1.00 at N, and you pay $P^{(N)}/P^{(N+1)}$ at $N+1$. You have synthesized a contract signed today for a loan from N to $N+1$—a forward rate! Thus,

$$F_t^{(N\to N+1)} = \text{ Forward rate at } t \text{ for } N \to N+1 = \frac{P_t^{(N)}}{P_t^{(N+1)}},$$

and of course

$$f_t^{(N\to N+1)} = p_t^{(N)} - p_t^{(N+1)}. \tag{19.3}$$

People sometimes identify forward rates by the initial date, $f_t^{(N)}$, and sometimes by the ending date, $f_t^{(N+1)}$. I use the arrow notation when I want to be really clear about dating a return.

Forward rates have the lovely property that you can always express a bond price as its discounted present value using forward rates,

$$\begin{aligned} p_t^{(N)} &= p_t^{(N)} - p_t^{(N-1)} + p_t^{(N-1)} - p_t^{(N-2)} - \cdots - p_t^{(2)} - p_t^{(1)} + p_t^{(1)} \\ &= -f_t^{(N-1\to N)} - f_t^{(N-2\to N-1)} - \cdots - f_t^{(1\to 2)} - y_t^{(1)} \end{aligned}$$

$y_t^{(1)} = f_t^{(0\to 1)}$ of course), so

$$p_t^{(N)} = -\sum_{j=0}^{N-1} f_t^{(j\to j+1)};$$

$$P_t^{(N)} = \left(\prod_{j=0}^{N-1} F_t^{(j\to j+1)}\right)^{-1}.$$

Intuitively, the price today must be equal to the present value of the payoff at rates you can lock in today.

In continuous time, we can define the instantaneous forward rate

$$f(N,t) = -\frac{1}{P}\frac{\partial P(N,t)}{\partial N} = -\frac{\partial p(N_t)}{\partial N}. \tag{19.4}$$

Then, forward rates have the same property that you can express today's price as a discounted value using the forward rate,

$$p(N,t) = -\int_{x=0}^{N} f(x,t)\,dx$$

$$P(N,t) = e^{-\int_{x=0}^{N} f(x,t)\,dx}.$$

Equations (19.3) and (19.4) express forward rates as derivatives of the price versus maturity curve. Since yield is related to price, we can relate forward rates to the yield curve directly. Differentiating the definition of yield $y(N,t) = -p(N,t)/N$,

$$\frac{\partial y(N,t)}{\partial N} = \frac{1}{N^2}p(N,t) - \frac{1}{N}\frac{\partial p(N,t)}{\partial N} = -\frac{1}{N}y(N,t) + \frac{1}{N}f(N,t).$$

Thus,

$$f(N,t) = y(N,t) + N\frac{\partial y(N,t)}{\partial N}.$$

In the discrete case, (19.3) implies

$$f_t^{(N\to N+1)} = -Ny_t^{(N)} + (N+1)y_t^{(N+1)} = y_t^{(N+1)} + N\left(y_t^{(N+1)} - y_t^{(N)}\right).$$

Forward rates are above the yield curve if the yield curve is rising, and vice versa.

Swaps and Options

Swaps are an increasingly popular fixed-income instrument. The simplest example is a fixed-for-floating swap. Party A may have issued a 10-year fixed coupon bond. Party B may have issued a 10-year variable-rate bond—a bond that promises to pay the current one-year rate. (For example, if the current rate is 5%, the variable-rate issuer would pay $5 for every $100 of face value. A long-term variable-rate bond is the same thing as rolling over one-period debt.) They may be unhappy with these choices. For example, the fixed-rate payer may not want to be exposed to interest rate risk that the present value of his promised payments rises if interest rates decline. The variable-rate issuer may want to take on this interest rate risk, betting that rates will rise or to hedge other commitments. If they are unhappy with these choices, they can *swap* the payments. The fixed-rate issuer pays off the variable-rate coupons, and the variable-rate issuer pays off the fixed-rate coupons. Obviously, only the difference between fixed and variable rate actually changes hands.

Swapping the payments is much safer than swapping the bonds. If one party defaults, the other can drop out of the contract, losing the difference in price resulting from intermediate interest rate changes, but not losing the principal. For this reason, and because they match the patterns of cashflows that companies usually want to hedge, swaps have become very popular tools for managing interest rate risk. Foreign exchange swaps are also popular: Party A may swap dollar payments for party B's yen payments. Obviously, you do not need to have issued the underlying bonds to enter into a swap contract—you simply pay or receive the difference between the variable rate and the fixed rate each period.

The value of a pure floating-rate bond is always exactly one. The value of a fixed-rate bond varies. Swaps are set up so no money changes hands initially, and the fixed rate is calibrated so that the present value of the fixed payments is exactly one. Thus, the "swap rate" is the same thing as the yield on a comparable coupon bond.

Many fixed-income securities contain options, and explicit options on fixed-income securities are also popular. The simplest example is a call option. The issuer may have the right to buy the bonds back at a specified price. Typically, he will do this if interest rates fall a great deal, making a bond without this option more valuable. Home mortgages contain an interesting prepayment option: if interest rates decline, the homeowner can pay off the loan at face value, and refinance. Options on swaps also exist; you can buy the right to enter into a swap contract at a future date. Pricing all of these securities is one of the tasks of term structure modeling.

19.2 Yield Curve and Expectations Hypothesis

> The *expectations hypothesis* is three equivalent statements about the pattern of yields across maturity:
> 1. The N-period yield is the average of expected future one-period yields.
> 2. The forward rate equals the expected future spot rate.
> 3. The expected holding period returns are equal on bonds of all maturities.
>
> The expectations hypothesis is not quite the same thing as risk neutrality, since it ignores $1/2\sigma^2$ terms that arise when you move from logs to levels.

The *yield curve* is a plot of yields of zero-coupon bonds as a function of their maturity. Usually, long-term bond yields are higher than short-term bond yields—a *rising* yield curve. Sometimes short yields are higher than long yields—an *inverted* yield curve. The yield curve sometimes has humps or other shapes as well. The *expectations hypothesis* is the classic theory for understanding the shape of the yield curve.

More generally, we want to think about the evolution of yields—the expected value and conditional variance of next period's yields. This is obviously the central ingredient for portfolio theory, hedging, derivative pricing, and economic explanation.

We can state the expectations hypothesis in three mathematically equivalent forms:

1. The N-period yield is the average of expected future one-period yields

$$y_t^{(N)} = \frac{1}{N} E_t\big(y_t^{(1)} + y_{t+1}^{(1)} + y_{t+2}^{(1)} + \cdots + y_{t+N-1}^{(1)}\big)(+\text{ risk premium}). \quad (19.5)$$

2. The forward rate equals the expected future spot rate

$$f_t^{(N\to N+1)} = E_t\left(y_{t+N}^{(1)}\right)(+\text{ risk premium}). \quad (19.6)$$

3. The expected holding period returns are equal on bonds of all maturities

$$E_t(\text{hpr}_{t+1}^{(N)}) = y_t^{(1)}(+\text{ risk premium}). \quad (19.7)$$

(The risk premia in (19.5–19.7) are related, but not identical.)

You can see how the expectations hypothesis explains the shape of the yield curve. If the yield curve is upward sloping—long-term bond yields are higher than short-term bond yields—the expectations hypothesis says this is because short-term rates are expected to rise in the future.

You can view the expectations hypothesis as a response to a classic misconception. If long-term yields are 10% but short-term yields are 5%, an unsophisticated investor might think that long-term bonds are a better investment. The expectations hypothesis shows how this may not be true.

If short rates are expected to rise in the future, this means that you will roll over the short-term bonds at a really high rate, say 20%, giving the same long-term return as the high-yielding long term bond. Contrariwise, when the short-term interest rates rise in the future, long-term bond prices decline. Thus, the long-term bonds will only give a 5% rate of return for the first year.

You can see from the third statement that the expectations hypothesis is roughly the same as risk neutrality. If we had said that the expected *level* of returns was equal across maturities, that would be the same as risk neutrality. The expectations hypothesis specifies that the expected *log* return is equal across maturities. This is typically a close approximation to risk neutrality, but not the same thing. If returns are lognormal, then $E(R) = e^{E(r)+(1/2)\sigma^2(r)}$. If mean returns are about 10% or 0.1 and the standard deviation of returns is about 0.1, then $\frac{1}{2}\sigma^2$ is about 0.005, which is very small but not zero. We could easily specify risk neutrality in the third expression of the expectations hypothesis, but then it would not imply the other two; $\frac{1}{2}\sigma^2$ terms would crop up.

The intuition of the third form is clear: risk-neutral investors will adjust positions until the expected one-period returns are equal on all securities. Any two ways of getting money from t to $t+1$ must offer the same expected return. The second form adapts the same idea to the choice of locking in a forward contract versus waiting and borrowing and lending at the spot rate. Risk-neutral investors will load up on one or the other contract until the expected returns are the same. Any two ways of getting money from $t+N$ to $t+N+1$ must give the same expected return.

The first form reflects a choice between two ways of getting money from t to N. You can buy an N-period bond, or roll over N one-period bonds. Risk-neutral investors will choose one over the other strategy until the expected N-period return is the same.

The three forms are mathematically equivalent. If every way of getting money from t to $t+1$ gives the same expected return, then so must every way of getting money from $t+1$ to $t+2$, and, chaining these together, every way of getting money from t to $t+2$.

For example, let us show that forward rate = expected future spot rate implies the yield curve. Start by writing

$$f_t^{(N-1\to N)} = E_t\big(y_{t+N-1}^{(1)}\big).$$

Add these up over N,

$$f_t^{(0\to 1)} + f_t^{(1\to 2)} + \cdots + f_t^{(N-2\to N-1)} + f_t^{(N-1\to N)} \\ = E_t\big(y_t^{(1)} + y_{t+1}^{(1)} + y_{t+2}^{(1)} + \cdots + y_{t+N-1}^{(1)}\big).$$

The right-hand side is already what we are looking for. Write the left-hand side in terms of the definition of forward rates, remembering $P^{(0)} = 1$

so $p^{(0)} = 0$,

$$\begin{aligned} & f_t^{(0\to 1)} + f_t^{(1\to 2)} + \cdots + f_t^{(N-2\to N-1)} + f_t^{(N-1\to N)} \\ &\quad = \left(p_t^{(0)} - p_t^{(1)}\right) + \left(p_t^{(1)} - p_t^{(2)}\right) + \cdots + \left(p_t^{(N-1)} - p_t^{(N)}\right) \\ &\quad = -p_t^{(N)} = N y_t^{(N)}. \end{aligned}$$

You can show all three forms (19.5)–(19.7) are equivalent by following similar arguments.

It is common to add a constant *risk premium* and still refer to the resulting model as the expectations hypothesis, and I include a risk premium in parentheses to remind you of this idea. One end of each of the three statements does imply more risk than the other. A forward rate is known while the future spot rate is not. Long-term bond returns are more volatile than short-term bond returns. Rolling over short-term real bonds is a riskier long-term investment than buying a long-term real bond. If real rates are constant, and the bonds are nominal, then the converse can hold: short-term real rates can adapt to inflation, so rolling over short nominal bonds can be a safer long-term real investment than long-term nominal bonds. These risks will generate expected return premia if they covary with the discount factor, and our theory should reflect this fact.

If you allow an arbitrary, time-varying risk premium, the model is a tautology, of course. Thus, the entire content of the "expectations hypothesis" augmented with risk premia is in the restrictions on the risk premium. We will see that the constant risk premium model does not do that well empirically. One of the main points of term structure models is to quantify the size and movement over time in the risk premium.

19.3 Term Structure Models—A Discrete-Time Introduction

Term structure models specify the evolution of the short rate and potentially other state variables, and the prices of bonds of various maturities at any given time as a function of the short rate and other state variables. I examine a very simple example based on an AR(1) for the short rate and the expectations hypothesis, which gives a geometric pattern for the yield curve. A good way to generate term structure models is to write down a process for the discount factor, and then price bonds as the conditional mean of the discount factor. This procedure guarantees the absence of arbitrage. I give a very simple example of an AR(1) model for the log discount factor, which also results in geometric yield curves.

A natural place to start in modeling the term structure is to model yields statistically. You might run regressions of changes in yields on the levels of lagged yields, and derive a model of the mean and volatility of yield changes. You would likely start with a factor analysis of yield changes and express the covariance matrix of yields in terms of a few large factors that describe their common movement. The trouble with this approach is that you can quite easily reach a statistical representation of yields that implies an arbitrage opportunity, and you would not want to use such a statistical characterization for economic understanding of yields, for portfolio formation, or for derivative pricing. For example, a statistical analysis usually suggests that a first factor should be a "level" factor, in which all yields move up and down together. It turns out that this assumption violates arbitrage: the long-maturity yield must converge to a constant.[1]

How do you model yields without arbitrage? An obvious solution is to use the discount factor existence theorem: Write a statistical model for a positive discount factor, and find bond prices as the expectation of this discount factor. Such a model will be, by construction, arbitrage-free. Conversely, any arbitrage-free distribution of yields can be captured by some positive discount factor, so you do not lose any generality with this approach.

A Term Structure Model Based on the Expectations Hypothesis

We can use the expectations hypothesis to give the easiest example of a term structure model. This one does not start from a discount factor and so may not be arbitrage-free. It does quickly illustrate what we mean by a "term structure model."

Suppose the one-period yield follows an AR(1),

$$y^{(1)}_{t+1} - \delta = \rho(y^{(1)}_t - \delta) + \varepsilon_{t+1}.$$

Now, we can use the expectations hypothesis (19.5) to calculate yields on bonds of all maturities as a function of today's one-period yield,

$$\begin{aligned} y^{(2)}_t &= \frac{1}{2}E_t\left[y^{(1)}_t + y^{(1)}_{t+1}\right] \\ &= \frac{1}{2}\left[y^{(1)}_t + \delta + \rho(y^{(1)}_t - \delta)\right] \\ &= \delta + \frac{1+\rho}{2}(y^{(1)}_t - \delta). \end{aligned}$$

[1] More precisely, the long-term forward rate, if it exists, must never fall. Problem 7 guides you through a simple calculation. Dybvig, Ingersoll, and Ross (1996) derive the more general statement.

Continuing in this way,

$$\left(y_t^{(N)} - \delta\right) = \frac{1}{N}\frac{1-\rho^N}{1-\rho}(y_t^{(1)} - \delta). \tag{19.8}$$

You can see some issues that will recur throughout the term structure models. First, the model (19.8) can describe different yield curve shapes at different times. If the short rate is below its mean, then there is a smoothly upward sloping yield curve. Long-term bond yields are higher, as short rates are expected to increase in the future. If the short rate is above its mean, we get a smoothly inverted yield curve. This particular model cannot produce humps or other interesting shapes that we sometimes see in the term structure. Second, this model predicts no average slope of the term structure: $E(y_t^{(N)}) = E(y_t^{(1)}) = \delta$. In fact, the average term structure seems to slope up slightly and more complex models will reproduce this feature. Third, all bond yields move together in the model. If we were to stack the yields up in a VAR representation, it would be

$$\begin{aligned}
y_{t+1}^{(1)} - \delta &= \rho\left(y_t^{(1)} - \delta\right) + \varepsilon_{t+1},\\
y_{t+1}^{(2)} - \delta &= \rho\left(y_t^{(2)} - \delta\right) + \frac{1+\rho}{2}\varepsilon_{t+1},\\
&\vdots\\
y_{t+1}^{(N)} - \delta &= \rho\left(y_t^{(N)} - \delta\right) + \frac{1}{N}\frac{1-\rho^N}{1-\rho}\varepsilon_{t+1}.
\end{aligned}$$

(You can write the right-hand variable in terms of $y_t^{(1)}$ if you want—any one yield carries the same information as any other.) The *error terms are all the same.* We can add more factors to the short-rate process, to improve on this prediction, but most tractable term structure models maintain less factors than there are bonds, so some perfect factor structure is a common prediction of term structure models. Fourth, this model has a problem in that the short rate, following an AR(1), can be negative. Since people can always hold cash, nominal short rates are never negative, so we want to start with a short-rate process that does not have this feature. Fifth, this model shows no conditional heteroskedasticity—the conditional variance of yield changes is always the same. The term structure data show times of high and low volatility, and times of high yields and high yield spreads seem to track these changes in volatility. Modeling conditional volatility is crucially important for valuing term structure options.

With this simple model in hand, you can see some obvious directions for generalization. First, we will want more complex driving processes than an AR(1). For example, a hump shape in the conditionally expected short rate will result in a hump-shaped yield curve. If there are multiple state variables driving the short rate, then we will have multiple factors driving the yield curve which will also result in more interesting shapes. We also want processes that keep the short rate positive in all states of nature. Second, we will want to add some "market prices of risk"—some risk premia. This will allow us to get average yield curves to not be flat, and time-varying risk premia seem to be part of the yield data. We will also want to check that the market prices are reasonable, and in particular that there are no arbitrage opportunities.

The yield curve literature proceeds in exactly this way: specify a short-rate process and the risk premia, and find the prices of long-term bonds. The trick is to specify sufficiently complex assumptions to be interesting, but preserve our ability to solve the models.

The Simplest Discrete-Time Model

The simplest nontrivial model I can think of is to let the log of the discount factor follow an AR(1) with normally distributed shocks. I write the AR(1) for the log rather than the level in order to make sure the discount factor is positive, precluding arbitrage. Log discount factors are typically slightly negative, so I denote the unconditional mean $E(\ln m) = -\delta$

$$(\ln m_{t+1} + \delta) = \rho(\ln m_t + \delta) + \varepsilon_{t+1}.$$

In turn, you can think of this discount factor model as arising from a consumption-based power utility model with normal errors,

$$m_{t+1} = e^{-\delta}\left(\frac{C_{t+1}}{C_t}\right)^{\gamma},$$

$$c_{t+1} - c_t = \rho(c_t - c_{t-1}) + \varepsilon_{t+1}.$$

The term structure literature has only started to explore whether the empirically successful discount factor processes can be connected empirically back to macroeconomic events in this way.

From this discount factor, we can find bond prices and yields. This is easy because the conditional mean and variance of an AR(1) are easy to find. (I am following the strategy of solving the discount factor forward rather than solving the price backward.) We need

$$y_t^{(1)} = -p_t^{(1)} = -\ln E_t(e^{\ln m_{t+1}}),$$

$$y_t^{(2)} = -\frac{1}{2}p_t^{(2)} = -\frac{1}{2}\ln E_t(e^{\ln m_{t+1} + \ln m_{t+2}}),$$