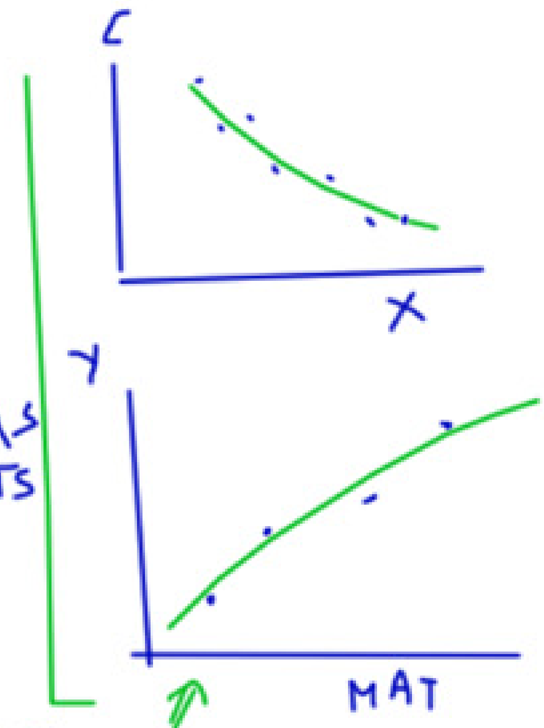
1 BONDS INTRO

- US: DEFAULT FREE, CASHFLOW KNOWN $P_t = E_t(M_{t+1} \cdot 1)$

- DATA PLOT
  - UP + DOWN WITH INFLATION
  - WITH RECESSIONS!
  - SPREAD HIGH IN RECESSION, LOW IN BOOMS
  - "LEVEL", "SLOPE" CAPTURE MOST MOVEMENTS
- "YIELD CURVE" / "FORWARD CURVE"



  - CAN SLOPE UP/DOWN
  - WHY? (ECONOMICS) - MODEL JUST LIKE BLACK-SCHOLES?
    - → PRICE ALL MATURITIES
    - → PRICE / HEDGE TS OPTIONS

Bonds 2. Definitions

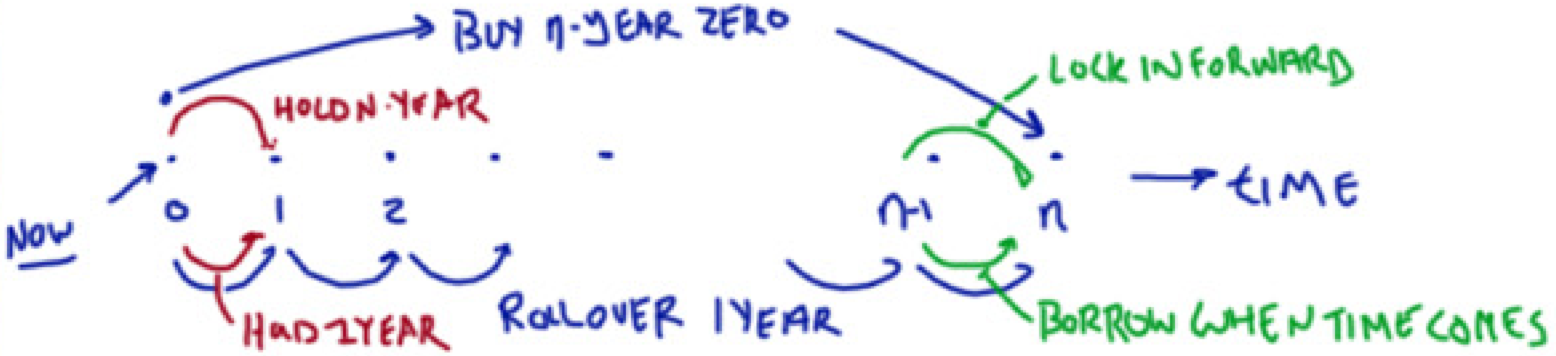
Zero-coupon bond → pays \$1 at $t+n$

Object	Idea	Example
Price $P_t^{(n)}$ (maturity; date)		0.9 (\$90 real world)
log price $p_t^{(n)} = \log(P_t^{(n)})$	"10% discount"	$-0.1 \approx \log(0.9)$
Yield $y_t^{(n)} = -\frac{1}{n} p_t^{(n)}$	"Average return" "Bond discount rate"	$p_t^{(2)} = -0.1$ "10% discount" = $y_t^{(2)} = +0.05$ "5%/year"
Forward $f_t^{(n)} = p_t^{(n-1)} - p_t^{(n)}$	"Rate you can lock in today, to borrow at n, pay back $n+1$"	$p_t^{(2)} = -0.1$, $p_t^{(1)} = -0.05$ → $f_t^{(2)} = 0.05$
Return $r_{t+1}^{(n)} = p_{t+1}^{(n-1)} - p_t^{(n)}$ (WARNING)		
Excess return $rx_{t+1}^{(n)} = r_{t+1}^{(n)} - y_t^{(1)}$	Warning: $\log(R-R^f) \neq \log(R) - \log(R^f)$	

③ Coupon bonds $P_t = P_t^{(1)} \cdot C + P_t^{(2)} \cdot C + \cdots + P_t^{(n)} \cdot [C+1]$ ← zeros like contingent claims

Bonds 3 Expectations Hypothesis

- Why does yield/forward curve slope?
- If $y_t^{(1)} = 0.1\%$, $y_t^{(10)} = 2\%$ is $y_t^{(10)}$ a better deal?
- EH ≈ "risk neutral", "random walk". No. $y^{(10)}$, $y^{(1)}$ give the same return in year 1, rates will rise.



$r_{0 \to n}$

A. $-p_t^{(n)} = E(y_t^{(1)} + y_{t+1}^{(1)} + \cdots + y_{t+n-1}^{(1)}) \Rightarrow y_t^{(n)} = E_t \frac{1}{n}(y_t^{(1)} + y_{t+1}^{(1)} + \cdots + y_{t+n-1}^{(1)})$ [+ risk prem.]

B. $y_t^{(1)} = E_t(r_{t+1}^{(n)})$; $E_t(rx_{t+1}^{(n)}) = 0 \ \forall n$ [+ risk premium]

C. $f_t^{(n)} = E_t(y_{t+n-1}^{(1)})$

Fact

A ⇔ B ⇔ C

3.1 EXPECTATIONS EXAMPLE

$p_t^{(1)} = -0.05$ $\Rightarrow$ $y_t^{(1)} = 0.05$

$p_t^{(2)} = -0.20$ $y_t^{(2)} = 0.10$

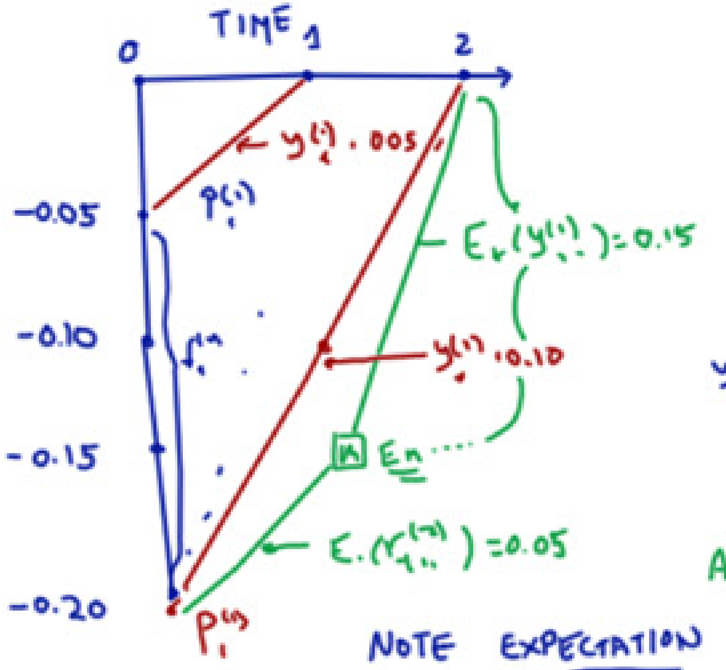
$f_t^{(1)} = 0.15$

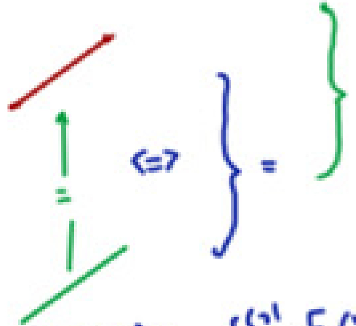
EH

$E_t(y_{t+1}^{(1)}) = 0.15$

$E_t(p_{t+1}^{(1)}) = -0.15$

$E_t(r_{t+1}^{(2)}) = 0.05$





$y_t^{(1)}, E_t(r_{t+1}^{(2)})$ $f_t^{(1)} = E_t(y_{t+1}^{(1)})$

$\Leftrightarrow y_t^{(2)} = \frac{1}{2} E(y_t^{(1)}, y_{t+1}^{(1)})$

ALL TRUE

NOTE EXPECTATION WILL END UP ELSEWHERE

4

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fx

$r^{EURO} = 5\%$, $r^{US} = 1\%$ → Buy?

EH → EURO ↓ 4% !

$E(S_{t+1}) - S_t = r_t^{EU} - r_t^{US}$ [+R.P.]

BOTTOM LINE

1) STARTING PLACE "ALL ER =" HAS SURPRISING IMPLICATIONS!

2) IDENTITY "Y MUST MOVE SO X DOES NOT"

EITHER $E_t(r_{t+n}^{(n)})$ IS HIGH OR $E_t(y_{t+1}^{(1)} - y_t^{(1)})$ IS HIGH

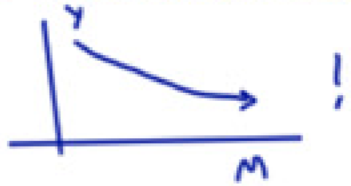
EH : WHAT MUST MOVE IF ER DOES NOT MOVE

BONDS 4 RISK PREMIUM INTRODUCTION

- WHICH IS RISKIER?
  - $r_{t+1}^{(2)}$ vs $y_t^{(1)}$
  - $y_t^{(n)}$ vs $y_t^{(1)}, y_{t+1}^{(1)} \ldots y_{t+n-1}^{(1)}$
  - WHAT IF REAL $y_t^{(1)}$ ARE CONSTANT, INFLATION VARIES?

INDEX BONDS ARE RISK FREE FOR LONG HORIZON INVESTORS. IGNORE PRICE CHANGES

REAL YIELD CURVE SHOULD SLOPE DOWN



BONDS 4 CONT'D

- $E_t(R^e_{t+1}) = \gamma\, cov_t(R^e_{t+1}, \Delta c_{t+1})$

  $= -\gamma\, n\, cov_t(y^{(n)}_{t+1}, \Delta c_{t+1})$

  ISSUE "MARKET PRICE OF INTEREST RATE RISK"

  DOES A SHOCK $y^{(n)}_{t+1}$ ↑ COME W. GOOD OR BAD Δc_{t+1}

  ↓ $E(R^e) < 0$!

- EMPIRICAL EVIDENCE [NO GOOD THEORY... YET]

- "EXPECTATIONS" - STRICT - NO R.P
  - RP IS SMALL, CONSTANT OVER TIME
  - NOT? RP VARIES OVER TIME

FX $\quad S_{t+1} - S_t = a + b(r^f_t - r^{*}_t) + \varepsilon_{t+1}$

SHOULD BE -1

IS $\quad 0 - +1$! "CARRY TRADE"

"UNCOVERED INTEREST PARITY" PUZZLE

RISK PREMIUM TO HOLD HIGH r CURRENCIES, BORROW LOW r

5

BONDS 5. FACTS: FAMA-BLISS

- LOOK AT DATA. GOOD!

- REGRESSIONS: (EXPECT? X_t)

$$rx^{(n)}_{t+1} = a + b\,(f^{(n)}_t - y^{(1)}_t) + \varepsilon_{t+1}$$

n	b	σ(b)	R^2
2	0.83	0.27	0.11
3	1.14	0.35	0.13
4	1.38	0.43	0.15
5	1.05	0.49	0.07

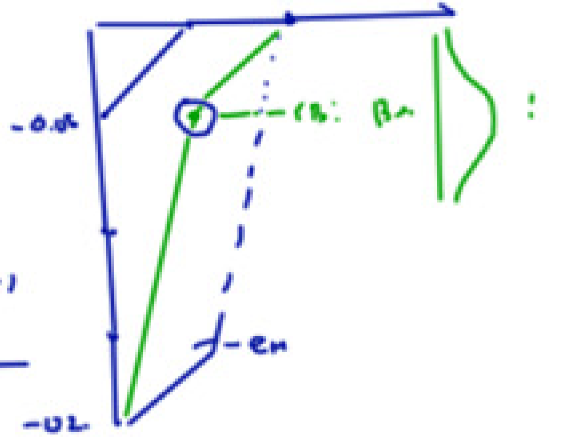
FORECAST 1 YEAR R ON n YEAR BONDS

(EXPECT?)

$$y^{(1)}_{t+n-1} - y^{(1)}_t = a + b\,(f^{(n)}_t - y^{(1)}_t) + \varepsilon_{t+1}$$

b	σ(b)	R^2
0.17	0.27	0.01
0.53	0.33	0.05
0.89	0.26	0.14
0.92	0.17	0.17

FORECAST ONE YEAR RATE n-1 YEARS FROM NOW

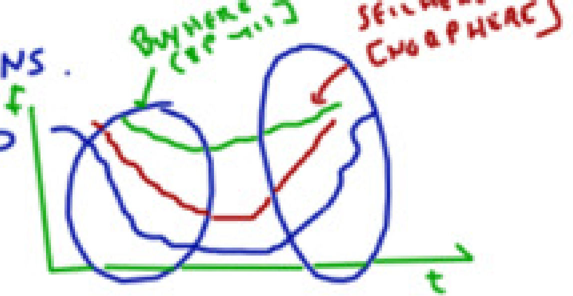


- $f^{(n)} - y^{(1)}$ ALMOST 1-1 TO ER

  (⌊⌈ GRAPH)

- $f^{(2)} - y^{(1)}$ ALMOST $\underline{0}$ TO $\Delta y^{(1)}$ ⇒ EH 100% WRONG! AT 1-YEAR
- $f^{(5)} - y^{(1)}$ ALMOST $\underline{1}$ TO $\Delta y^{(1)}_{t+4}$ ⇒ EH 100% RIGHT! AT 5-YEAR

- WHY DID EYES FAIL? • YOU GET PAID TO HOLD INTEREST RATE RISK IN RECESSIONS.



- MORE RECENT (AE) $rx^{(n)}_{t+1} = a + b(f^{(n)}_t - y^{(1)}_t) + \underline{c \cdot Z_t} + \varepsilon_{t+1}$ R^2 UP TO 0.44. CAN SENSE THE END

BONDS 6 STATISTICAL FACTOR MODELS

- IDEA : LIKE FF3F

$$y_t = \begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \\ y_t^{(3)} \\ y_t^{(4)} \\ y_t^{(5)} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}^{\text{LOADINGS}} \underbrace{\text{"level"}_t}_{\text{FACTORS}} + \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \text{"SLOPE"}_t + \text{SMALL } \varepsilon_t$$

$\Rightarrow$ HIGH R^2

LEVEL $= \frac{1}{5}\sum_{n=1} y_t^{(n)}$ (WEIGHTS) SLOPE $= y_t^{(5)} + y_t^{(4)} - (y_t^{(1)} + y_t^{(2)})$

- PRODUCE ? CHOOSE WEIGHTS + LOADINGS TO MAX R^2
  - WEIGHTS $\to$?
  - LOADINGS $\to$ OLS

- $\Sigma = \mathrm{cov}(y, y') = Q \Lambda Q' = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_r \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix} \begin{bmatrix} - & q_1 & - \\ - & q_2 & - \\ - & q_r & - \end{bmatrix}$

6

$QQ' = Q'Q = I$

FACTORS $x_t = Q' y_t \Rightarrow \mathrm{COV}(x, x') = Q' \Sigma Q = \Lambda$

$[q_i]$ ARE WEIGHTS $x_t^i = [- \; q_i \; -] \begin{bmatrix} | \\ y_t \\ | \end{bmatrix}$

FACTORS ARE UNCORRELATED

FACTOR MODEL $y_t = Q x_t = [q_1] x_t^1 + [q_2] x_t^2 \cdots + [q_5] x_t^5$

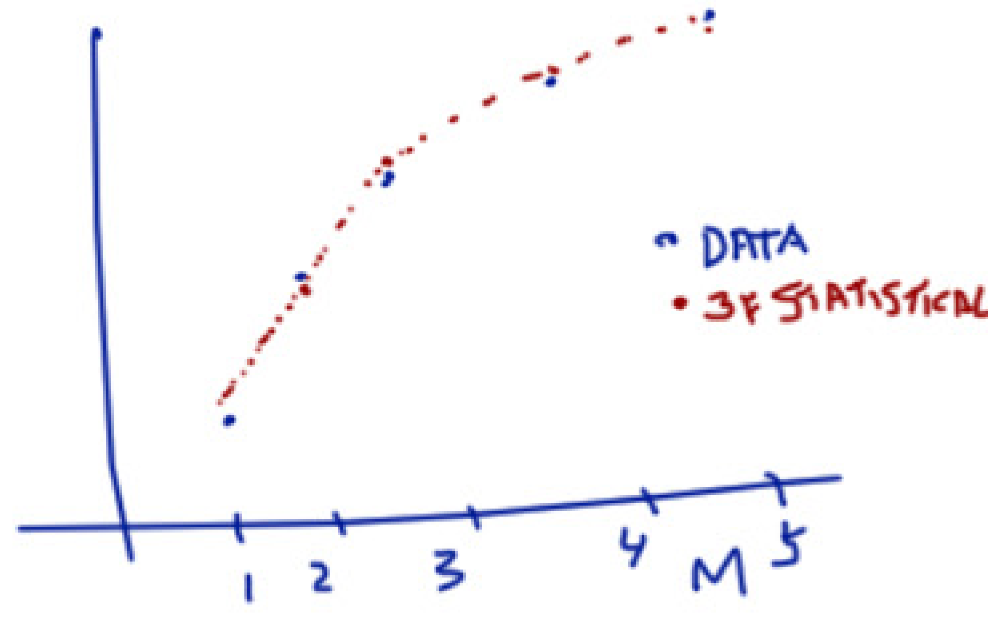
$[q_i]$ ARE LOADINGS

$$y_t = \underbrace{[q_1]}_{\text{biggest } \sigma^2} x_t^1 + [q_2] x_t^2 + \underbrace{[q_3] x_t^3 \cdots [q_5] x_t^5}_{\varepsilon_t}$$

- LEAVE OUT SOME FACTORS ?
  - $\to$ LOADINGS ARE OLS
  - $\to$ EACH x_t MAX R^2 !

6.1 STATISTICAL FACTOR MODEL IN DATA

- BONDS
- FF25



7 TERM STRUCTURE MODEL WITH EXPECTATIONS HYPOTHESIS

- SHOWS IDEA
  - NEED TO GO BEYOND INCLUDED BONDS
  - "ARBITRAGE"

- "SHORT RATE PROCESS"

$$(y^{(1)}_{t+1}-\delta) = \phi(y^{(1)}_t - \delta) + \varepsilon_{t+1}$$

- FIND OTHERS BY EH

$$f^{(2)}_t = E_t y^{(1)}_{t+1} = \delta + \phi(y^{(1)}_t-\delta)$$

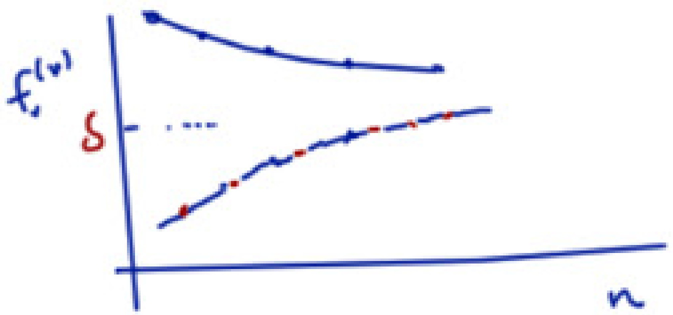
$$f^{(3)}_t = E_t(y^{(1)}_{t+2}) = \delta + \phi^2(y^{(1)}_t-\delta)$$

$$f^{(4)}_t = E_t(y^{(1)}_{t+3}) = \delta + \phi^3(y^{(1)}_t-\delta)$$

THEN

$$y^{(n)}_t = \frac{1}{n}(y^{(1)}_t + f^{(2)}_t + \dots f^{(n)}_t) = \delta + (1+\phi+\phi^2 + \dots \phi^{n-1})(y^{(1)}_t-\delta)$$

- AN EXACT 1-FACTOR MODEL!
- SHAPES



- MORE FACTORS?

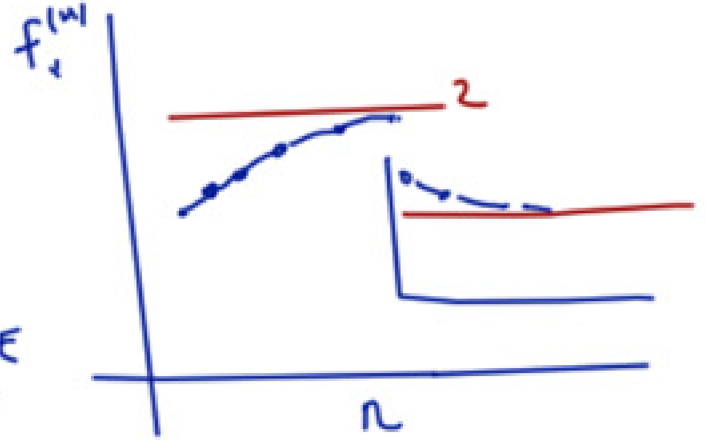
SHORT RATE PROCESS

$$\begin{cases} z_t = z_{t-1} + \varepsilon_t \\ x_t = \phi x_{t-1} + \varepsilon_t \\ y^{(1)}_t = z_t + x_t \end{cases}$$

$\rightarrow$ $f^{(2)}_t = E_t y^{(1)}_{t+1} = z_t + \phi x_t$ $\rightarrow$ LEVEL+SLOPE

$f^{(3)}_t = E_t y^{(1)}_{t+2} = z_t + \phi^2 x_t$

...



REAL MODEL? LIKE THIS BUT MODEL M_{t+1}, $P^{(n)}_t = E_t(M_{t+1} \cdots M_{t+n})$

Bonds 8 Discrete Time Vasicek

• Result

$$y^{(1)}_{t+1} - \delta = \phi(y^{(1)}_t - \delta) + \varepsilon_{t+1}$$

$$f^{(2)}_t = \delta + \phi(y^{(1)}_t - \delta) - \left[\tfrac{1}{2} + \lambda\right]\sigma^2_\varepsilon$$

$$f^{(3)}_t = \delta + \phi^2(y^{(1)}_t - \delta) - \left[\tfrac{1}{2}(1+\phi)^2 + \lambda(1+\phi)\right]\sigma^2_\varepsilon$$

$$f^{(4)}_t = \delta + \phi^3(y^{(1)}_t - \delta) - \left[\tfrac{1}{2}(1+\phi+\phi^2)^2 + \lambda(1+\phi+\phi^2)\right]\sigma^2_\varepsilon$$

$$f^{(n)}_t = \delta + \underbrace{\phi^{n-1}(y^{(1)}_t - \delta)}_{\text{Expectations}} - \underbrace{\left[\tfrac{1}{2}\left(\frac{1-\phi^{n-1}}{1-\phi}\right)^2 + \lambda\left(\frac{1-\phi^{n-1}}{1-\phi}\right)\right]\sigma^2_\varepsilon}_{\text{Risk premium} + \frac{1}{2}\sigma^2 \text{ terms}}$$

$$y^{(n)}_t = \frac{1}{n}\left\{y^{(1)}_t + f^{(2)}_t + \dots + f^{(n-1)}_t\right\} ; \quad p^{(n)}_t = -n y^{(n)}_t \quad \dots$$

• Derivation

$$\begin{cases} x_{t+1} - \delta = \phi(x_t - \delta) + \varepsilon_{t+1} \\ \ln M_{t+1} = m_{t+1} = -x_t - \frac{1}{2}\lambda^2\sigma^2_\varepsilon - \lambda\varepsilon_{t+1} \end{cases}$$

- A time series model of M_{t+1}
- x_t "state variable" shifts $E_t(M_{t+1})$
- $\lambda\varepsilon_{t+1}$: generates $cov(M_{t+1}, R_{t+1})$ risk premium

$$P^{(n)}_t = E_t(M_{t+1}M_{t+2}\cdots M_{t+n})$$

• Approach 1 : Brute force

$$P^{(1)}_t = E_t M_{t+1} = E_t e^{m_{t+1}} = E_t\, e^{-x_t - \frac{1}{2}\lambda^2\sigma^2_\varepsilon - \lambda\varepsilon_{t+1}} = e^{-x_t}$$

$$p^{(1)}_t = -x_t \quad y^{(1)}_t = x_t\ !$$

- "Bond prices reveal latent state variables"
- Can start with $y^{(1)}_t = x_t$, then check model produces $y^{(1)}_{t+1}$

$$P^{(2)}_t = E_t(M_{t+1}M_{t+2}) = E_t(M_{t+1}P^{(1)}_{t+1}) = E_t\, e^{-x_t - \frac{1}{2}\lambda^2\sigma^2_\varepsilon - \lambda\varepsilon_{t+1} \overbrace{-x_{t+1}}}$$

$$= E_t\, e^{-x_t - \frac{1}{2}\lambda^2\sigma^2_\varepsilon - \lambda\varepsilon_{t+1} - \delta - \phi(x_t - \delta) - \varepsilon_{t+1}}$$

$$= E_t\, e^{-2\delta - (1+\phi)(x_t - \delta) - \frac{1}{2}\lambda^2\sigma^2_\varepsilon - (1+\lambda)\varepsilon_{t+1}}$$

$$= E_t\, e^{\dots}$$

....

$$p^{(2)}_t = -2\delta - (1+\phi)(x_t - \delta) + \left(\tfrac{1}{2} + \lambda\right)\sigma^2_\varepsilon$$

$$f^{(2)}_t = p^{(1)}_t - p^{(2)}_t$$

BONDS 9: OTHER APPROACHES

- "SOLVE" DISCOUNT FACTOR + INTEGRATE"

$$P_t^{(2)} = E_t(M_{t+1}M_{t+2}) = E_t\Big[e^{-x_t-\frac{1}{2}\lambda^2\sigma_\varepsilon^2-\lambda\varepsilon_{t+1}\;-\overbrace{[\delta+\phi(x_t-\delta)+\varepsilon_{t+1}]}^{x_{t+1}}-\frac{1}{2}\lambda^2\sigma_\varepsilon^2-\lambda\varepsilon_{t+2}}\Big]\dots$$

HAVE FUN!

- "DERIVE A DIFFERENCE / DIFFERENTIAL EQUATION FOR PRICE AS FUNCTION OF STATE VARIABLES"
  - GUESS $p_t^{(n)} = A_n - B_n(x_t-\delta)$
  - USE PRICING TO $p_t^{(n-1)} \to p_t^{(n)}$ $\quad P_t^{(n)} = E_t(M_{t+1}P_{t+1}^{(n-1)})$

$$A_n - B_n(x_t-\delta) = \log E_t \exp\Big[\overbrace{-x_t-\tfrac{1}{2}\lambda^2\sigma_\varepsilon^2-\lambda\varepsilon_{t+1}}^{m_{t+1}} + \overbrace{A_{n-1}-B_{n-1}(x_{t+1}-\delta)}^{p_{t+1}^{(n-1)}}\Big]$$

$x_{t+1}-\delta = \phi(x_t-\delta)+\varepsilon_{t+1}$

$$= -\delta + A_{n-1} - (1+B_{n-1}\phi)(x_t-\delta) + \left(B_{n-1}\lambda+\tfrac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2$$

$\Rightarrow B_n = 1+\phi B_{n-1}$; $A_n = -\delta + A_{n-1} + \left(B_{n-1}\lambda + \frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2$

$B_0 = 0 \quad B_1 = 1 \quad B_2 = 1+\phi$ $\qquad A_0 = 0 \quad A_1 = -\delta \quad A_2 = -2\delta + \left(\lambda + \frac{1}{2}\right)\sigma_\varepsilon^2$

$B_3 = 1+\phi+\phi^2 \dots$ $\qquad A_3 = -3\delta + \left[(1+\phi)\lambda + \frac{1}{2}(1+\phi)^2\right]\sigma_\varepsilon^2 \dots$

$$B_n = \frac{1-\phi^n}{1-\phi}$$

2

- RISK-NEUTRAL APPROACH

$$f_t^{(n)} = (1-\phi^{n-1})\,\delta + \phi^{n-1}y_t^{(1)} - \left[\frac{1}{2}\left(\frac{1-\phi^{n-1}}{1-\phi}\right)^2 + \lambda\left(\frac{1-\phi^{n-1}}{1-\phi}\right)\right]\sigma_\varepsilon^2$$

$$f_t^{(n)} = (1-\phi^{n-1})\underbrace{\left[\delta - \frac{\lambda}{1-\phi}\sigma_\varepsilon^2\right]}_{\delta^*} + \phi^{n-1}y_t^{(1)} - \left[\frac{1}{2}\left(\frac{1-\phi^{n-1}}{1-\phi}\right)^2\right]\sigma_\varepsilon^2$$

- THE CROSS-SECTIONAL MODEL IS THE SAME AS A MODEL WITH $\lambda=0$ RISK NEUTRAL + DISTORTED MEAN $\delta^* = \delta - \frac{\sigma_\varepsilon^2}{1-\phi}\lambda$

$$x_{t+1} = \delta^* + \phi(x_t-\delta^*) + \varepsilon_{t+1}$$

$$m_{t+1} = -x_t$$

$$y_t^{(1)} = x_t$$

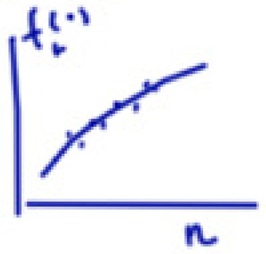
$$p_t^{(2)} = \log e^{-x_t-x_{t+1}} = \log e^{-x_t-\delta^*-\phi(x_t-\delta^*)-\varepsilon_{t+1}}$$

$$= -2\delta^* - (1+\phi)(x_t-\delta^*) + \tfrac{1}{2}\sigma_\varepsilon^2$$

ALGEBRA IS EASIER!

10. COMMENTS ON SINGLE FACTOR VASICEK

- LIKE EH : "SHORT RATE PROCESS" (FACTOR PROCESS) · "SINGLE FACTOR MODEL"
  - "SHORT RATE PROCESS" → HOW YIELDS MOVE OVER TIME / FORECASTS
  - "SINGLE FACTOR MODEL" → CROSS SECTION



- RISK PREMIUM IN MODEL. SIGN OF λ : CAN GO EITHER WAY
- "EXPECTATIONS HYPOTHESIS ≠ RISK NEUTRALITY" $\frac{1}{2}\sigma^2$. SMALL $\sigma \approx 0.01$
- EXAMPLE GRAPH
- RISK-NEUTRAL : λ IS NOT NEEDED TO DESCRIBE CROSS SECTION.
  λ CONNECTS CROSS SECTION ("TOO HIGH" δ") WITH TIME SERIES FORECASTS.
  → "RISK NEUTRAL" USED FOR MANY PRICING CASES.
  → BE CAREFUL!! $\delta^* \neq \delta$! [$\phi^* \neq \phi$!]
- $M_{t+1} > 0$ → "ARBITRAGE FREE" MODEL PRICE $P = E(M_{t+1} X^c_{t+1})$
  W/O ARBITRAGE (AT LEAST)
  (BLACK-SCHOLES FOR BOND OPTIONS)

- TO DO
  - MORE FACTORS : $X_t = \begin{bmatrix} x_t^1 \\ x_t^2 \\ \vdots \\ x_t^k \end{bmatrix}$ ← VECTOR
- TIME-VARYING RISK PREMIUMS [FAMA-BLISS]

$$X_{t+1} - \mu = \Phi(X_t - \mu) + \varepsilon_{t+1} \qquad E(\varepsilon_{t+1}\varepsilon_{t+1}') = V$$

$$\log M_{t+1} = m_{t+1} = -\delta' X_t - \frac{1}{2}\lambda_t' V^{-1} \lambda_t - \lambda_t' \varepsilon_{t+1}$$

$$\lambda_t = \lambda_0 + \Lambda_1 X_t$$

$$\Rightarrow \mu^* = \mu - V\lambda_0 \qquad \phi^* = \phi - V\Lambda_1$$

$$f_t^{(n)} = \dots + \delta_1' \underbrace{\phi^{*\,n-1} X_t}_{\text{FIT SHAPE}}$$

- MORE... EASIER IN CONT. TIME.

3

BONDS II: CONTINUOUS TIME TERM STRUCTURE

- SAME IDEAS!
- SIMPLE CASE: VASICEK.

$[r = x]$

$$\begin{cases} \boxed{dr_t = \phi(\bar{r} - r_t)\,dt + \sigma_r\,dz_t = \mu_r(r_t)\,dt + \sigma_r\,dz_t} \\ \dfrac{d\Lambda_t}{\Lambda_t} = -r_t\,dt - \sigma_\Lambda\,dz_t \end{cases}$$

$f(n,t) := -\dfrac{\partial p(n,t)}{\partial n}$

RESULT

$$\boxed{f(n,t) - f_t^{(1)} = \underbrace{\bar{r} + e^{-\phi n}(r_t - \bar{r})}_{\text{expectations}} - \underbrace{\left[\frac{1}{2}\frac{\sigma_r^2}{\phi^2}(1 - e^{-\phi n})^2 + \sigma_\Lambda\frac{\sigma_r}{\phi}(1 - e^{-\phi n})\right]}_{\text{risk premium, } \frac{1}{2}\sigma^2}}$$

APPROACH
- GUESS $P(n,r)$. FIND PDE FOR $P(n,r)$ FROM INSTANTANEOUS PRICING.

- PRICING → DE

$$E_t(dR_t) - r_t\,dt = -E_t\left(\frac{d\Lambda}{\Lambda}\,dR_t\right)$$

$$E_t\left(\frac{dP}{P}\right) - \frac{1}{P}\frac{\partial P(n,t)}{\partial n} - r_t\,dt = -E_t\left(\frac{d\Lambda}{\Lambda}\frac{dP}{P}\right) \quad (A)$$

DEFINITION OF $P(n,r)$
TIME TO GO
NOT TIME OF MATURITY

- Ito $dP(n,r) = \left[\dfrac{\partial P}{\partial r}\mu_r + \dfrac{1}{2}\dfrac{\partial^2 P}{\partial r^2}\sigma_r^2\right]dt + \dfrac{\partial P}{\partial r}\sigma_r\,dz_t \quad (B)$

4

(B) → (A)

$$\frac{\partial P}{\partial r}\mu_r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma_r^2 - \frac{\partial P}{\partial n} - rP = \frac{\partial P}{\partial r}\sigma_r\sigma_\Lambda$$

$$\boxed{\frac{\partial P(n,r)}{\partial n} = (\mu_r - \sigma_r\sigma_\Lambda)\frac{\partial P(n,r)}{\partial r} + \frac{1}{2}\frac{\partial^2 P(n,r)}{\partial r^2}\sigma_r^2 - rP(n,r)}$$

μ^*

- BOND PDE. LIKE BLACK-SCHOLES PDE. $P(0,r) = 1$, SOLVE (NUMERICALLY) FOR n. → $P(n=0,r)$

- GUESS $P(n,r) = e^{A(n) - B(n)r}$ FIND $A(n)$ $B(n)$

$$\frac{\partial P}{\partial r} = -B(n)P \;;\; \frac{\partial^2 P}{\partial r^2} = B(n)^2 P \;;\; \frac{\partial P}{\partial n} = [A'(n) - B'(n)r]P$$

$$\Rightarrow A'(n) - B'(n)r = -[\phi(\bar{r} - r) - \sigma_r\sigma_\Lambda]B(n) + \frac{1}{2}\sigma_r^2 B(n)^2 - r$$

($\phi\bar{r}$)

$B'(n) = 1 + \phi B(n)$ $\quad A'(n) = -[\phi\bar{r} - \sigma_r\sigma_\Lambda]B(n) + \frac{1}{2}\sigma_r^2 B(n)^2$

- TURNED STOCHASTIC DIFFERENTIAL EQUATION + EXPECTATION $P(n,r_t) = E_t[\Lambda_{t+n}/\Lambda_t]$ [HARD, BUT OK NUMERICALLY BY MONTE CARLO, SIMULATE Λ_t FORWARD]
  INTO PARTIAL DIFFERENTIAL EQUATION [HARD, BUT OK NUMERICALLY]
  INTO ORDINARY DIFFERENTIAL EQUATION [REALLY FAST IF NUMERICAL; EASY HERE BY HAND]

- $\dfrac{dB(n)}{dn} = 1 + \phi B(n)$; $\displaystyle\int \frac{1}{1+\phi B}\,dB = \int dn$; $-\dfrac{1}{\phi}\log(1+\phi B) = n$; $B_n = \dfrac{1}{\phi}(1 - e^{-\phi n}) \iff B(n) = \dfrac{1-\phi^n}{1-\phi}$! $A(n)$ similarly.

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12. CONTINUOUS TIME DISCUSSION

- "PAITOR PROCESS" (TIME) + "FACTOR MODEL" (CROSS SECTION)
- EXPECTATIONS + RISK PREMIUM, $\frac{1}{2}\sigma^2$ TERMS
- RISK NEUTRAL TRICK $\bar{r}^* = \bar{r} - \frac{\sigma_m \sigma_r}{\phi}$

  $\Rightarrow f(n,r) = \bar{r}^* + e^{-\phi n}(r - \bar{r}^*) - \frac{1}{2}\frac{\sigma_r^2}{\phi^2}(1 - e^{-\phi n})^2$

  DEEPER $\underset{\mu_r^*}{\mu_r - \sigma_r \sigma_m}$; $\underset{\phi \bar{r}^*}{\phi\bar{r} - \sigma_r \sigma_m}$ ONLY PLACE σ_m ENTERS

- SIMPLIFIES ALGEBRA, BUT DON'T FIT μ_r^* OR $\bar{r}^*$ FROM TIME SERIES!
  $AB\pi$

- MORE REALISTIC: SAME STRUCTURE, IDEAS.
  - KEEP $r_t > 0$; SOME CHANGING VOLATILITY: CIR

  $dr_t = \phi(\bar{r} - r_t) + \sqrt{r}\,\sigma_r dz_t$

(5)

- MULTIPLE FACTORS: $r \to x$, A VECTOR
- "MULTIFACTOR AFFINE"

  $dx_t = \Phi(\bar{x} - x_t)dt + \Sigma dw_t$

  $r_t = \delta_0 + \delta_1' x_t$

  $\frac{d\Lambda}{\Lambda} = -r_t dt - b_t' dw_t$

  $dw_i = \sqrt{\alpha_i + \beta_i' x_t}\, dz_t$

  $\rightarrow$ STILL $P(n,t) = e^{-A(n) - B(n)' x_t}$

  $A(n)$ $B(n)$ FAST ODE

- "UNSPANNED VARIABLES"

  x_t, STOCHASTIC VOL. THAT

  IS NOT REVEALED BY BOND PRICES.

λ MORE IMPORTANT

$\lambda \leftarrow$ ECONOMY MORE IMPORTANT

- POINT OF ALL THIS

  (WHICH MODEL—WHICH PURPOSE)

  - CROSS SECTION – PRICE WEIRD BONDS, OPTIONS
  - FIND PRICING ANOMALIES, "LIQUIDITY"
  - FIND TRADING STRATEGIES
  - FORECAST RATES
  - INTERPRET CURVE $\rightarrow$ EXPECTED REAL, π, PREMIUM?