PORTFOLIO THEORY I: CLASSIC

- WHAT TO BUY? CLASSIC: RETURNS $(N\times 1)$ R, MODEL

  $E(R)$, $\mathrm{cov}(R,R') = \Sigma$, $U(\cdot)$, $R^p_{t+1} = w'R$. BEST w?

- LOGNORMAL IID POWER UTILITY

- $\max_{\{c_t, w_t\}} \int_{t=0}^{\infty} u(c_t)\,dt$ s.t. W_0,

  $dW_t = W_t[r^f dt + w_t'(dR_t - r^f dt)] - c_t dt$

  $dR_t = \mu\, dt + \sigma\, dz_t$ $\quad \sigma\sigma' = \Sigma$

(LIKE BLACK-SCHOLES)

$\rightarrow dW_t = W_t[r^f dt + w_t(\mu - r^f)dt + w'\sigma dz] - c_t dt$

$\{c_t, w_t\}$??

- SOLVE BY DYNAMIC PROGRAMMING.

$$V(W_t) = \max_{\{c_t, w_t\}} \int_{s=0}^{\infty} e^{-\rho s} u(c_{t+s})\,ds \text{ s.t.} \ldots = \max_{\{c_t, w_t\}} u(c_t)\Delta + E[e^{-\rho\Delta} V(W_{t+\Delta})] \text{ s.t.} \ldots$$

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$$0 = \max u(c_t)\Delta + E_t[e^{-\rho\Delta} V(W_{t+\Delta}) - V(W_t)]$$

$$0 = \max u(c_t) - \rho V(W_t) + V_W(W_t) E_t(dW_t) + \frac{1}{2} V_{WW}\, dW_t^2$$

$$* \quad 0 = \max_{\{c_t, w_t\}} u(c_t) - \rho V(W_t) + V_W\{W[r^f + w_t'(\mu - r^f)] - c_t\} + \frac{1}{2} W^2 V_{WW} w_t' \Sigma w_t$$

① $\partial/\partial w_t$: $w_t^* = -\frac{V_W}{W \cdot V_{WW}} \Sigma^{-1}(\mu - r^f)$ $\quad -\frac{V_W}{W V_{WW}} = \frac{1}{\gamma} \rightarrow$ $\boxed{w_t^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r^f)}$

② $\partial/\partial c_t$: $u'(c_t) = V_W$

- INVESTOR HOLDS A CONSTANTLY REBALANCED MEAN-VARIANCE EFFICIENT PORTFOLIO

  (REMINDER: $\min \mathrm{VAR}(w'R^e)$ s.t. $w'E(R^e) = E$

  $w'\Sigma w - \lambda w'(\mu - r^f) \rightarrow w = \lambda \Sigma^{-1}(\mu - r^f)$!

- FACT: IF $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ THEN $V(W) = k \cdot \frac{W^{1-\gamma}}{1-\gamma}$ $\rightarrow \gamma$ IS V CURVATURE

  $\rightarrow$ FRACTION w_t IS CONSTANT! (NO t, NOW)

  $\rightarrow \int_0^T \cdot V(W_T)$ ALSO HORIZON DOES NOT MATTER

  $\rightarrow c_t^{-\gamma} = k W_t^{-\gamma}$

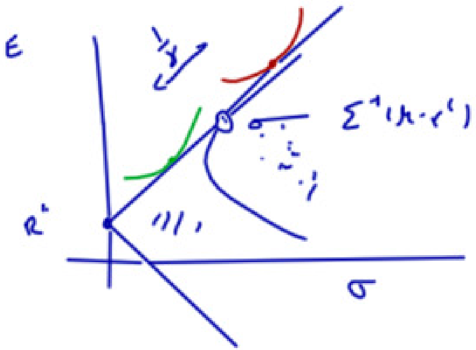
  $\boxed{c_t = k^{-\frac{1}{\gamma}} W_t}$

  CONSUME CONSTANT FRACTION OF WEALTH!

- HOW?

$$\gamma \Rightarrow \frac{c_t^{1-\gamma}}{1-\gamma} - \rho V - V_W W[r^f + w^{*\prime}(\mu - r^f) - c_t] + \frac{1}{2} V_{WW} w^{*\prime} \Sigma^{-1} w^* = 0$$

PORTFOLIO THEORY 2 MEAN-VARIANCE



- "TWO-FUND" THEOREM
- $\Sigma^{-1}(\mu - r^f)$ SAME FOR ALL – NO MGMT, TAILORED PORTFOLIOS
- ONLY E, β, Σ MATTER NOT NAMES & STYLES.
- THINK OF PORTFOLIO NOT ASSETS IN ISOLATION
- STILL TRUE. WIDELY IGNORED.

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- RELATIVE TO MARKET. IF ALL LIKE THIS WITH DIFFERENT γ

$$w^i = \frac{\gamma^m}{\gamma^i} w_i \qquad R^{pi} = R^f + \frac{\gamma^m}{\gamma^i} R^{em} \qquad w^m, R^{em} \text{ ON MVF}$$

- IF ... CAPM $E(R^{ei}) = R^f + \beta_{im} E(R^{em})$

- ALPHAS. WRITE STATISTICAL MODEL [WHETHER OR NOT α THEM =, CAPM]

$$R^{ei}_t = \alpha_i + \beta_{im} R^{em}_t + \varepsilon^i_t \qquad i = 1 \dots N$$

$$R^p_t = R^f + w_m R^{em}_t + w'_\varepsilon (R^e - \beta R^{em}) = R^f + w_m R^m + w'_\varepsilon(\alpha + \varepsilon) \qquad E(\varepsilon\varepsilon') = \Sigma$$

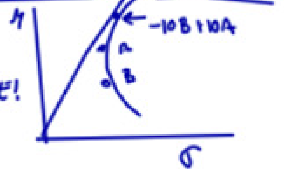
$$\rightarrow w_m = \frac{1}{\gamma} \frac{E(R^{em})}{\sigma^2} \qquad w_\varepsilon = \frac{1}{\gamma}\Sigma^{-1}\alpha$$

← DIVERSIFY αS. MAY HAVE $w > 0$ WITH $\alpha < 0$!

← SMALL α_i WITH SMALL Σ_{ii} GREAT IF YOU CAN LEVER!

- NUMBERS.
  - $\frac{1}{2} \cdot \frac{0.08}{(0.20)^2} = 1$! ✓
  - $C_t \sim h^{\frac{1}{\gamma}} W_t \rightarrow \sigma\left(\frac{dc}{c}\right) = \sigma\left(\frac{dw}{w}\right) = 0.20$ X

- $\Sigma^{-1}\mu$ IS VERY UNSTABLE.
- NO JOB?
- $E_t(R)$ VARIES! $\sigma_t(R)$
- CAPM FALSE!



PORTFOLIOS 3: MERTON :

$$\max_{\{C_t, w_t\}} \int_0^\infty u(C_t)\,dt \quad \text{s.t. } W_0,$$

JOB, BENEFITS, ETC.

$$dW_t = W_t\left[r_t^f\,dt + w_t'(dR_t - r_t^f\,dt)\right] + (y_t - C_t)\,dt$$

TIME-VARYING $E_t(R)$, Σ_t ! $\leftrightarrow$ $dR_t = \mu(X_t)\,dt + \sigma(X_t)\,dz_t \qquad \sigma\sigma' = \Sigma_t$

$$r_t^f = r^f(X_t) \qquad y_t = y(X_t)$$

"STATE VARIABLES" $\rightarrow$ $dX_t = \mu_x(X_t)\,dt + \sigma_x(X_t)\,dz_t$

- $V(W_t, X_t) = \max \int_{s=0}^\infty e^{-\rho s} u(C_{t+s})\,ds \quad \text{s.t. } W_t, X_t$
- $0 = \max u(C_t) \cdots + V_x dX_t + V_{wx}\,dW dX + \frac{1}{2}V_{xx}\,dX^2$

$$\underline{w_t'\,dR_t\,dX_t}$$

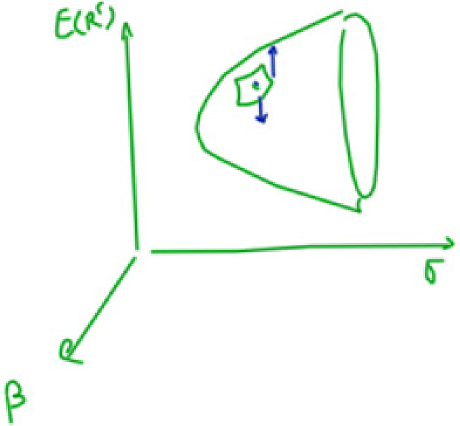
$$\Rightarrow \boxed{w_t = -\frac{1}{\gamma}\Sigma_t^{-1}E_t(dR_t - r_t^f\,dt) - \frac{\eta}{\gamma}\beta_{dR_t, dX_t}} \Leftarrow \mathrm{cov}(dx\,dx')^{-1}\mathrm{cov}(dx, dR)$$

"MARKET-TIMING" "STATE VARIABLE HEDGING"

$\gamma = -\frac{V_{ww}W}{V_w} \qquad \eta = \frac{V_{wx}}{V_w}$ = "AVERSION TO STATE VARIABLE RISK"

NOT UTILITY. JOB, TIME ETC.

$(\gamma_t, \gamma(W_t, X_t)!) \qquad (\eta_t, \eta(W_t, X_t)!)$



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- 3 FUND THEOREM
- R^m NOT MVF! HFE!
- $R_t^{ei} = R^f + \frac{\gamma^m}{\gamma^i} R_t^{em} + \frac{1}{\gamma^i}(\eta^i - \eta^m) R_t^{ex}$ — $R^{x} = \beta_{x,R}\, R^e$
- $E(R^{ei}) = \gamma^m \mathrm{cov}(R^{ei}; R^{em}) + \eta^m \mathrm{cov}(R^{ei}; dx)$

$\rightarrow$ $\eta^i = 0$? BUY; WRITE INSURANCE

$\rightarrow$ $\eta^i > \eta^m$? SELL; BUY INSURANCE!

$\rightarrow$ $\eta^i \neq 0, \eta^m = 0$? BUY FREE INSURANCE

NON-PRICED FACTORS! SHORT YOUR (.) BUY

PORTFOLIOS 4: MERTON EXAMPLES

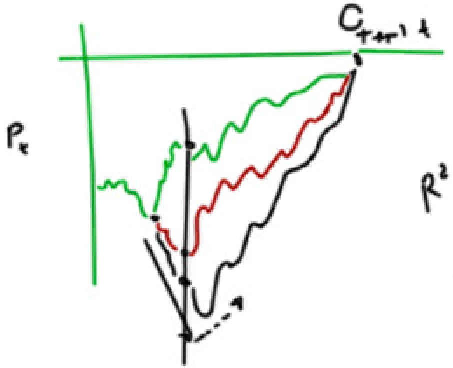
· THING: $w_t = \frac{1}{\gamma}\frac{E_t(R^e_{t+1})}{\sigma_t^2} = \frac{1}{\gamma}\frac{a+b\cdot(D/P)_t}{\sigma_t^2}$

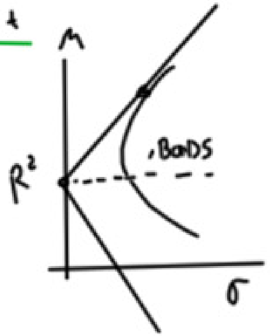
$$R^e_{t+1} = a + b\cdot(D/P)_t + \varepsilon_{t+1}$$

$\sigma(a+b\,D/P) \simeq 5\%$ $E(R) \simeq 7\%$

(GRAPH):   HUGE! TRUSTWORTHY?

- HEDGING.





→ HUGE! 1PERIOD μ, Σ IS 100% WRONG:

PORTFOLIOS 5: COMMENTS

• BIG PICTURE

$$\max E\,U(C_{t+1})$$

$$C_{t+1} = W_{t+1} = [R^f + w R^e_{t+1}]W_t$$

$$\max_w E\,U[(R^f + w R^e_{t+1})W_t]$$

$$\partial/\partial w \quad E[U'[\underbrace{W_{t+1}}_{C_{t+1}}]R^e_{t+1}] = 0$$

$$\rightarrow \quad E[U'(C_{t+1})R^e_{t+1}] = 0$$

· BEFORE: Fix $\{C_{t+1}\} \rightarrow E(R^e)$ "WHAT ER, σ MUST BE TO INDUCE INVESTOR TO JUST HOLD MARKET"

· NOW: Fix $E(R^e), \sigma \rightarrow C_{t+1}$!

- Conundrum
  - Average investor holds market
  - Average α is zero - α is zero sum game

    relative to market

  - Cannot all rebalance
  - → Different? "Smarter"? $\Sigma^{-1}\mu$?

    → Merton - lots of risks! More important than α:

    Express as differences.

  - $\Sigma^{-1}\mu$ is horribly unstable

    → Differences, economic function.

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- Paradox. Merton hedge not used.

  – $\eta = \frac{V_{Wy}}{V_W}$ Too nervous? Hard only numerical

  – Bond example. Look at it wrong?

- Discount factors

  $E_t(R_{t+1})\ \Sigma_t \rightarrow M_{t+1}$

  max $E\,U(C_{t+1})$ s.t. $E(M_{t+1}C_{t+1}) = W_0$

  $U'(C_{t+1}) = \lambda M_{t+1}$ $\quad C_{t+1}^{-\gamma} = \lambda M_{t+1}$

  $C_{t+1} = U'^{-1}(\lambda M_{t+1})$ $\quad C_{t+1} = (\lambda M_{t+1})^{-\frac{1}{\gamma}}$

  Options? Dynamic portfolio?

  Separate final payoff from financial engineering.

- Portfolio theory: guide, parables, avoid mistakes. Quantitative answers: