MEAN-VARIANCE FRONTIER - I. CLASSIC APPROACH

BIG PICTURE: $P=E(mx) \leftrightarrow E(R^i)=R^f+\beta_i \cdot \lambda \leftrightarrow$ MVF; MOMENTS → STATES → π^*

NOW IN COMPLETE MARKET, STATE SPACE, ∞ ASSETS, PRETTIER

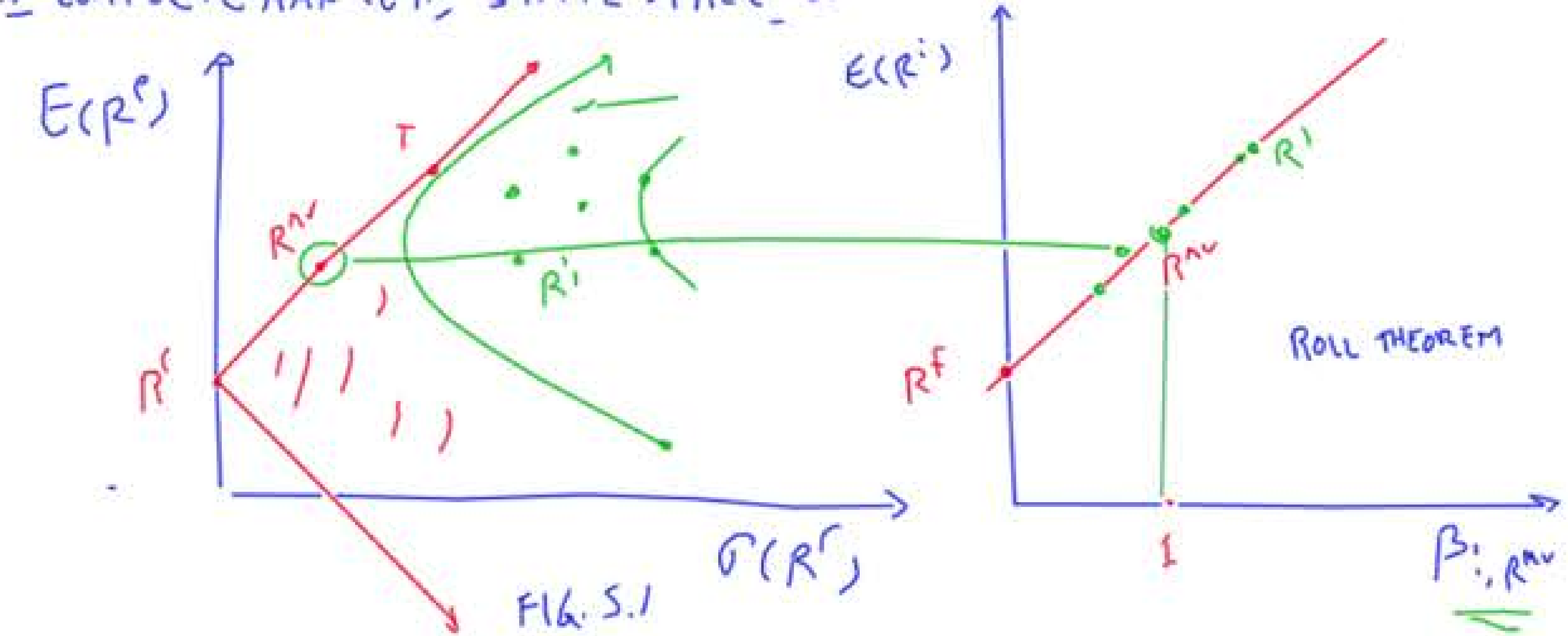


FIG. S.1

ALGEBRA

$$R^e = \begin{bmatrix} R^{e1} \\ \vdots \\ \vdots \\ R^{eN} \end{bmatrix} \qquad R^{ep} = w'R^e; \quad \min \sigma^2(R^{ep}) \text{ S.T. } E(R^{ep}) = E$$

$$\min w'\Sigma w \text{ S.T. } w'\mu = E \qquad \left(\text{NOT: } w'1 = 1 \quad \sum_{i=1}^{N} w_i = 1\right)$$

$$\frac{\partial}{\partial w}\left[\frac{\partial}{\partial w_i}\right]: \quad \boxed{\Sigma w = \delta \mu} \; *$$

$$\boxed{w = \delta \Sigma^{-1} \mu}$$ ← ALL PORTFOLIO OPTIMIZERS

$$[\delta]\begin{bmatrix} \Sigma^{-1} \end{bmatrix}\begin{bmatrix} \mu \end{bmatrix} = \begin{bmatrix} w \end{bmatrix}$$

$$R^{emv} = \delta \mu' \Sigma^{-1} R^e$$

$$E(R^{emv}) = \delta \mu' \Sigma^{-1} \mu \qquad \sigma^2(R^{emv}) = \delta^2 \mu' \Sigma^{-1} \mu$$

$$\boxed{\frac{E(R^{emv})}{\sigma(R^{emv})} = \sqrt{\mu' \Sigma^{-1} \mu}}$$ ← MAX SHARPE RATIO

A. ONE-FUND THEOREM (R^*)

B. ROLL THEOREM

$* \rightarrow$ $$COV(R^e, \underbrace{R^{e\prime} w}_{R^{emv}}) = \delta E(R^e)$$

$$E(R^{ei}) = \beta_{i,R^{emv}} \cdot \lambda_{mv}$$

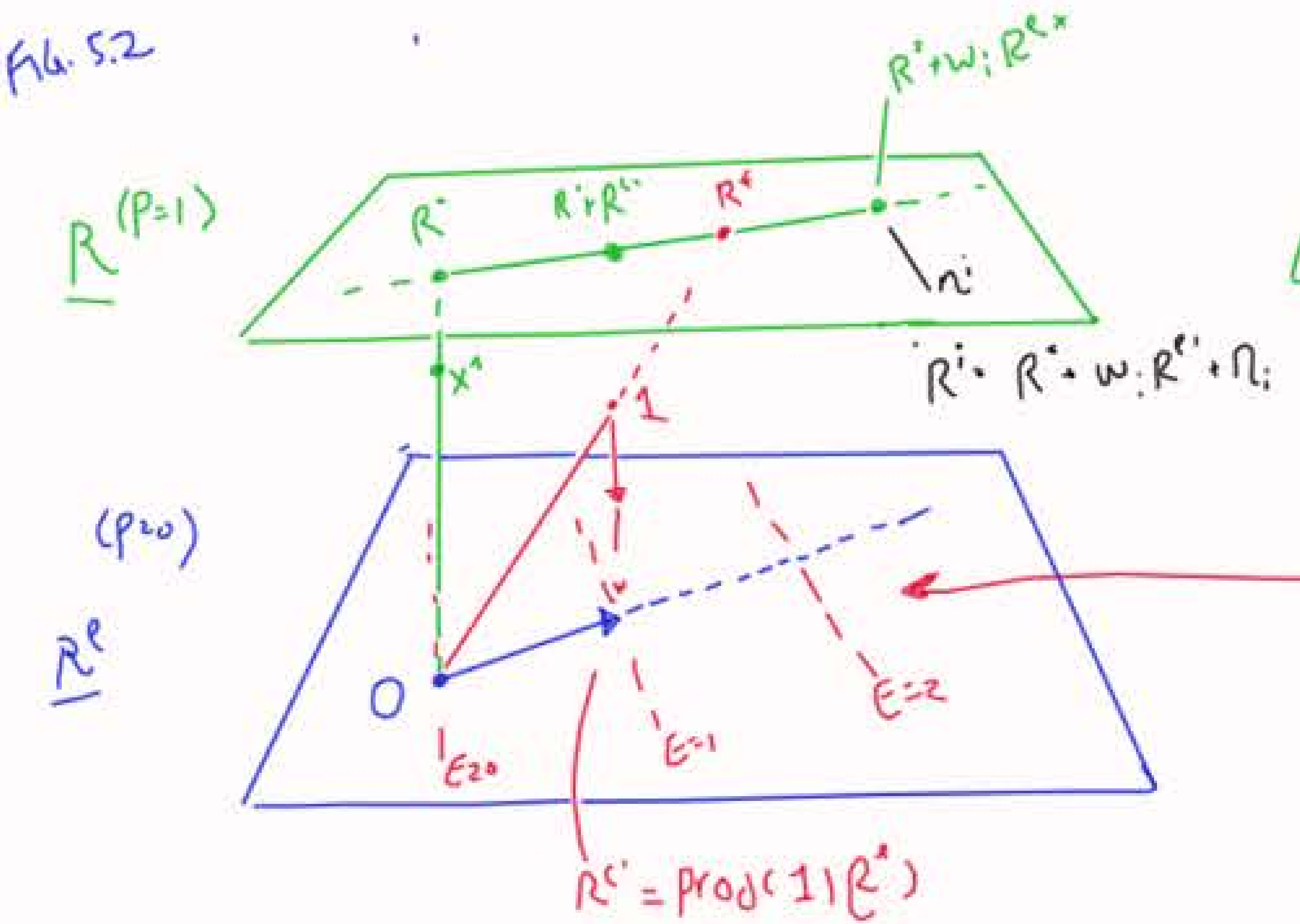
A SINGLE-β REPRESENTATION WITH R^{emv} REFERENCE ↔ R^{emv} ON MVF, $R^{emv} \neq 0$

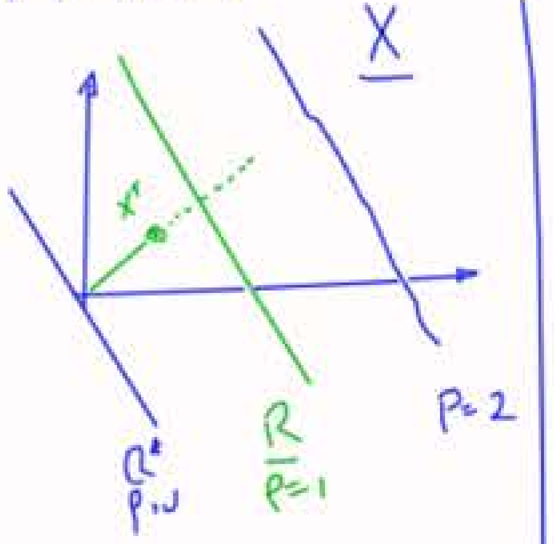
C. NOT (YET) R^{emv} IS THE OPTIMAL PORTFOLIO FOR ANY INVESTOR OR MARKET

D. IFF $R^{MKT} = R^{MV} \iff$ CAPM

2. STATE-SPACE [HANSEN-RICHARD] APPROACH TO MVF.

FIG. 5.2





1) ORTHOGONAL DECOMPOSITION

ANY RETURN $R^i = R^* + w_i R^{e*} + n_i$; $E(R^* R^{e*}) = E(R^* n_i) = E(R^{e*} n_i) = 0$

$E(n_i) = 0$

PROOF DEFINE n_i, SHOW ⊥

2) MEAN VARIABLE FRONTIER $R^{MV} = R^* + w \cdot R^{e*}$ → TWO FUND THEOREM

PROOF $E(R^i) = R^* + w_i E(R^{e*})$ $E(R^{i2}) = E(R^{*2}) + w_i^2 E(R^{e*2}) + \cancel{E(n_i^2)}$

(LOOP → $\exists\, x^* \in \underline{X}$ ...)

1) $R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})}$. $1 = E(x^* R) \Rightarrow E(R^{*2}) = E(R^* R)$

EXAMPLE: $R^* = \frac{1' E(RR')^{-1} R}{1' E(RR')^{-1} 1}$

2) $R^{e*} = \text{PROJ}(1 | \underline{R}^e)$

→ x^* CARRIES PRICE INFORMATION (1) $p(x) = E(x^* x)$

R^{e*} CARRIES MEAN INFORMATION $E(R^e) = E(R^{e*} R^e)$

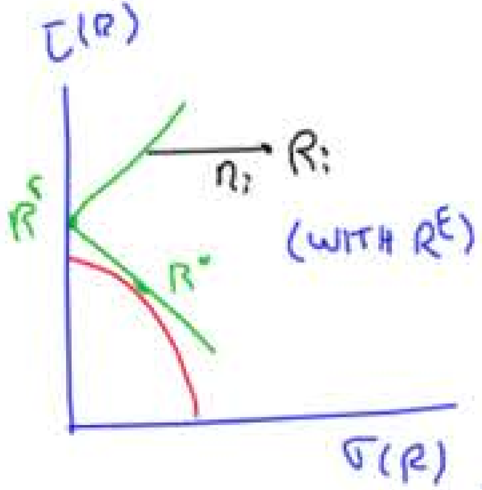
$p(x) = E(mx) = E(\text{proj}(m|\underline{X}) \cdot x) = E(x^* x)$

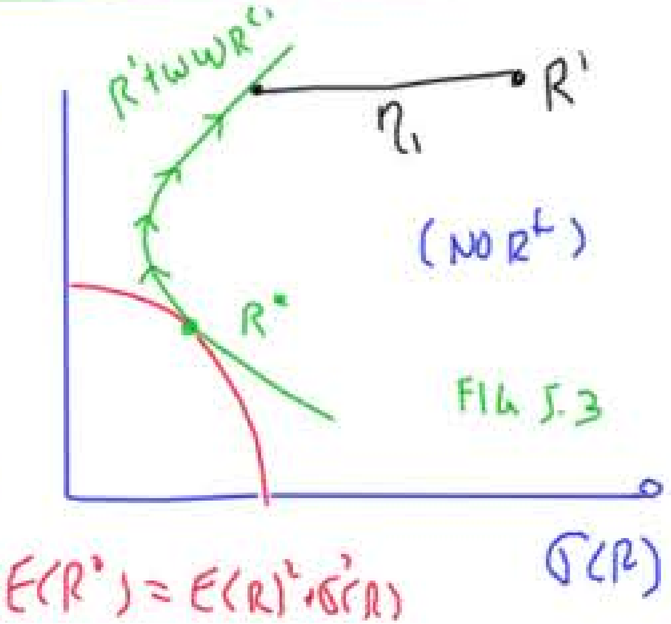
$E(R^e) = E(1 R^e) = E(\text{proj}(1|\underline{R}^e) \cdot R^e) = E(R^{e*} R^e)$

· EXAMPLE: $R^e = \begin{bmatrix} R^{e1} \\ \vdots \\ R^{en} \end{bmatrix}$, $R^{e*} = E(R^e)' E(R^e R^{e\prime})^{-1} R^e$

⟺ $x^* = p' E(xx')^{-1} x$

3. Comparing frontiers

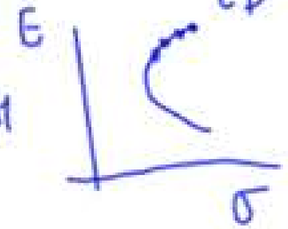




- R^* is the "smallest" minimum second moment return
- $E(\eta_i) = 0$, + $E(\eta_i \cdot R^*) = E(\eta_i \cdot R^{e*}) = 0$ regression residual, "idiosyncratic"
- "Only systematic component of risk is priced"
- R^* is not anyone's or the market portfolio!
- R^* is the return on the marginal utility mimicking portfolio. $R^* = \frac{x^*}{p(x^*)}$ $\quad x^* = proj(m \mid \underline{X})$

$m = \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$ goes down when $R^{portfolio}$, $\frac{c_{t+1}}{c_t}$ go up

R^* is negatively correlated with



- New $R^{emv} = w \cdot R^{e*}$; $R^{e*} = E(R^e)' E(R^e R^{e\prime})^{-1} R^e$. Old $R^{emv} = w\, E(R^e)' \Sigma^{-1} R^e$

Can do new with Σ too.

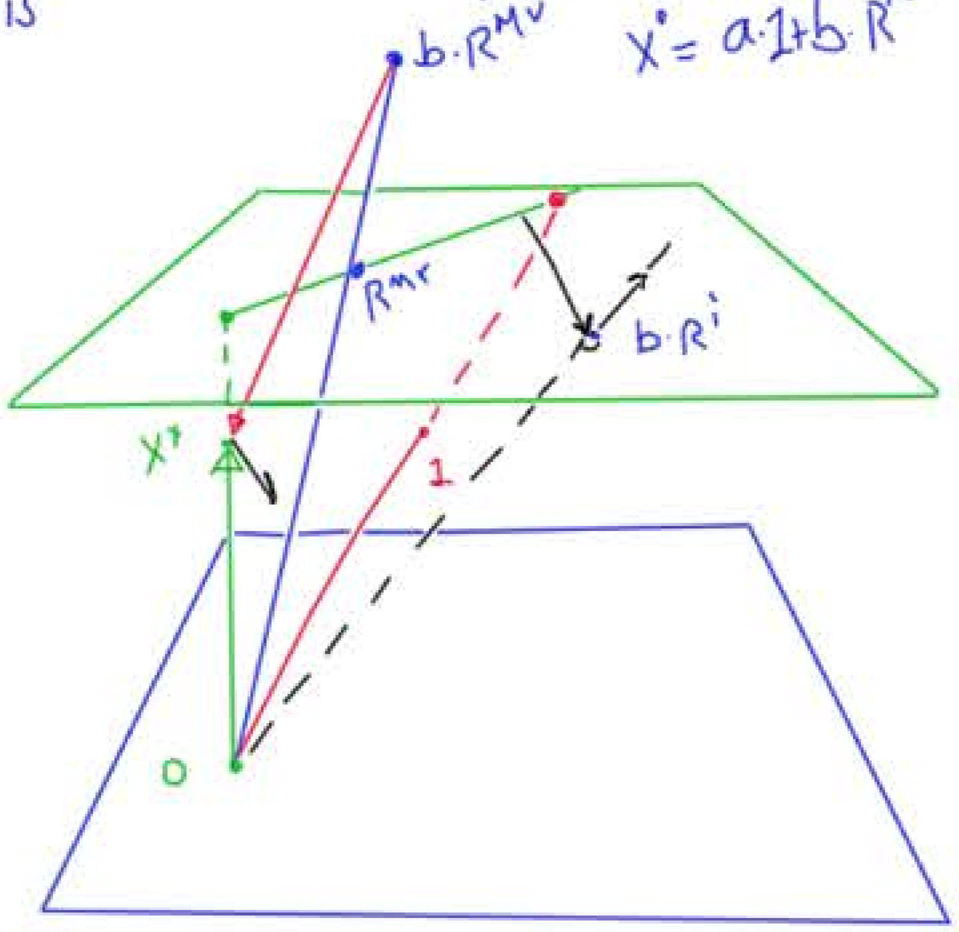
4. Roll theorem

(with R^f)

R^{mv} on MVF, $R^{mv} \neq R^f \iff$ ($\exists a,b:$) $m = a + bR^{mv}$ prices all $R \in \underline{X}$

i.e. $1 = E(m \cdot R)$

$\iff E(R^i) = R^f + \beta_{i,R^{mv}} \cdot \lambda_{mv}$



To β: $m = a + bf \iff E(R^{ei}) = \beta_{i,f} \cdot \lambda_f$

Excess return algebra

R^{emv} on MVF $\neq 0 \Rightarrow$

$R^{emv} = w \cdot R^{e*}$

Try $m = 1 - R^{e*} + \eta$

$E(m R^{ei}) =$

$= E[(1 - R^{e*} + \eta)(R^{e*} + \eta_i)]$

$= \underbrace{E(R^{e*}) - E(R^{e*2})}_{0} + E(\eta \eta_i)$

$\Rightarrow$ iff $\underline{\underline{\eta = 0}}$

EXISTENCE + EQUIVALENCE – SUMMARY + IMPLICATIONS

A. REPRESENTATIONS

CONTINUOUS TIME $\Rightarrow$

LOOP: M EXISTS

$\boxed{P = E(mx)} \leftrightarrow \boxed{P = E^{*}(x)/R^{f}}$

$\boxed{E(R^{ei}) = \beta_i \lambda} \longleftrightarrow \boxed{R^{MV} \text{ on MVF}}$

$\{\Sigma$ NON-SINGULAR $+ R^{MV}\}$

i) $P = E(mx)$ ISN'T MUCH. $M = f(\text{DATA})$ IS EVERYTHING.

ii) USE THE REPRESENTATION THAT WORKS BEST.

iii) KNOW HOW TO TRANSLATE

2. REPRESENTATION + HISTORY

B. HISTORY OF (i)

- FAMA (1971) "EFFICIENCY"

$R^{e}_{t+1} = a + b\, z_t + \varepsilon_{t+1}$

VS "IRRATIONAL" — IS π RIGHT?

FAMA "JOINT HYPOTHESIS"

1) $0 = E_t(M_{t+1} R^{e}_{t+1}) \rightarrow \underline{\underline{M_{t+1} R^{e}_{t+1}}} = a + b z_t + \varepsilon_{t+1}$

2) $P_t = \sum_s \pi_s \cdot M_s \, x_s$

ALWAYS TOGETHER

3) $\forall \pi$, NO ARBITRAGE $\rightarrow \underline{\exists} m : P$ is "RIGHT" $= E(mx)$

ASSET MARKET DATA ALONE CANNOT SETTLE "RATIONALITY"; "EFFICIENCY"

- ROLL

TESTS $E(R^{ei}) = \beta_{im} \lambda_m$

ASSUMPTIONS … ?

NO $\leftrightarrow$ R^{em} ON MVF

$E(R^{ei}) = \beta_{im} \lambda$ IS TRIVIAL $\quad R^{MV}$ IDENTITY IS KEY

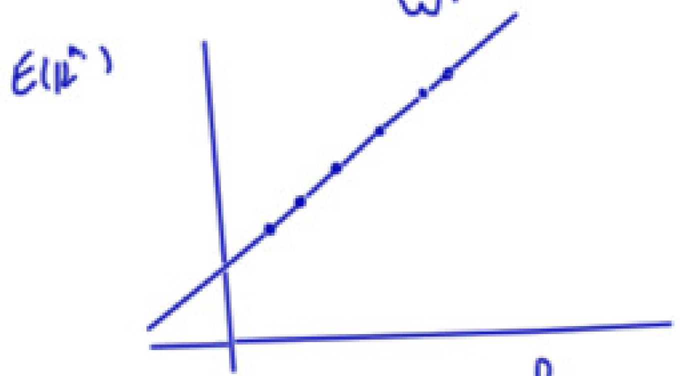
USING MARKET "PROXY" IS NOT INNOCUOUS. "CAPM" (WEALTH PORTFOLIO) UNTESTABLE

3. FISHING. $E(R)$, $E(RR')$ ETC. PROBABILITIES?

HOLDS FOR ANY PROBABILITIES;

SAMPLE $x^* = p' E(xx')^{-1} x \rightarrow$ SAMPLE P.

SAMPLE $R^{emv} = \underbrace{E(R^e)' \Sigma^{-1}}_{w'} R^e \rightarrow$ PERFECT FIT



$$R^{ei}_t = \alpha_i + \overset{\beta_{i,mv}}{\beta_{i,mv}} R^{emv}_t + \varepsilon_t$$

CT POST MV

- THERE IS A PORTFOLIO WHICH PERFECTLY PRICES ALL ASSETS IN SAMPLE
- + WORKS HORRIBLY OUT OF SAMPLE (SHOULDA BOUGHT GOOGLE)

$\Rightarrow$ RULES OF THE GAME ARE VITAL; CONSTRAINTS ON FACTOR FISHING.

EXAMPLE: FAMA+FRENCH. FACT MUCK ART LITTLE SCIENCE

4. MIMICKING PORTFOLIO THEOREM + FISHING

$P = E(mx)$ SAY $m = \beta\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$

$$P = E\left(\left(\underbrace{\text{proj}(m|\underline{x})}_{x^*} + \varepsilon\right)\cdot x\right) = E\left(\text{proj}(m|\underline{x})\cdot x\right) = E(x^* x)$$

$\rightarrow \exists$ a PORTFOLIO WHICH PRICES JUST AS WELL.

$\rightarrow \exists$ PORTFOLIOS THAT PRICE A LOT BETTER IN SAMPLE!

$\rightarrow$ IF C IS NOT MEASURED WELL x^* IS BETTER

$\Rightarrow$ COMPARING FIT OF $\underline{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}}$ + FF3F IS SILLY. IS FF3F x^* FOR $\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$ IS NOT SILLY

$\Rightarrow$ USING x^* (FF3F) IS RIGHT FOR MOST PRACTICAL QUESTIONS

$\Rightarrow$ USING x^* CANNOT ANSWER "RATIONALITY"

$\Rightarrow$ THEOREMS ARE IMPORTANT! GUIDE "WHY ARE WE DOING THIS"

FINDING RIGHT MODEL FOR EACH QUESTION!

CONDITIONING INFORMATION

- REALLY $P_t = E_t(M_{t+1} X_{t+1})$

$$E_t(R^e_{t+1}) = \beta_t \cdot \lambda_t$$

- $E_t(R_{t+1})$, β_t, λ_t VARY A LOT ; $R_{t+1} = a + b\left(\frac{D}{P_t}\right) + \varepsilon_{t+1}$

  ARCH / GARCH ETC.

- → COMPLEX MODELS OF E_t, β_t, σ_t …

- INFORMATION SETS. $E(R \mid I)$; $\mathrm{cov}(R^i, R^j \mid I)$ WHAT'S I?

  AGENTS ; MARKET ; ALL VARIABLES WE SEE ;

  VARIABLES WE INCLUDE ; UNCONDITIONAL

1. CONDITIONING DOWN.

$$0 = E(M_{t+1} R^e_{t+1} \mid \text{AGENT INFO}_t) \qquad M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

$E(\cdot) \rightarrow 0 = E(M_{t+1} R^e_{t+1} \mid \text{OUR INFO}_t)$

$\rightarrow 0 = E(M_{t+1} R^e_{t+1})$ UNCONDITIONAL AVERAGE $\Rightarrow$ $\boxed{0 = E(M_{t+1} R^e_{t+1}) \text{ IS A VALID IMPLICATION OF AGENT FOC ASSET PRICING}}$

$P = E(mx)$ "CONDITIONS DOWN"

$$E(E(x \mid I)) = E(x) \quad \text{LAW OF ITERATED EXPECTATIONS}$$

$$E_t(R^{ei}_{t+1}) = \beta_t \cdot \lambda_t$$

$E(R^{ei}_{t+1}) = E(\beta_{i,t} \lambda_t) \ldots ?\Rightarrow$ DOES NOT EASILY CONDITION DOWN.

2. INSTRUMENTS + MANAGED PORTFOLIOS

$$0 = E_t(M_{t+1} R^e_{t+1}) \to E(M_{t+1} R^e_{t+1}) = 0$$

POWER?

SUFFICIENCY

$$0 = E(M_{t+1} R^e_{t+1} z_t) \ \forall z_t \in I_t \Leftrightarrow 0 = E_t(M_{t+1} R^e_{t+1})$$

$$0 = E_t(M_{t+1} R^e_{t+1} z_t) \Rightarrow$$

$$0 = E(M_{t+1} R^e_{t+1} z_t)$$

(1) $E(M_{t+1} R^e_{t+1}) = 0$?

(2) $M_{t+1} R^e_{t+1} = a + b z_t + \varepsilon_{t+1} \to b \Rightarrow$ more (ALSO A TEST)

[GMM $0 = E(M_{t+1} R^e_{t+1} z_t)$?]

(3) $0 = E(M_{t+1} \underbrace{R^e_{t+1} z_t}_{R^p_{t+1}})$

TIME SERIES = CROSS SECTION

REGRESSIONS = PORTFOLIOS

"MANAGED PORTFOLIO THEOREM"

Ex: FF 25 ; USE HEDGE/MUTUAL FUNDS

3. CONDITIONAL + UNCONDITIONAL MODELS

PROBLEM PARAMETERS MAY VARY OVER TIME

"CONDITIONAL MODEL"

EXAMPLE: CONDITIONAL CAPM

$$E_t(R^{ei}_{t+1}) = \beta_{it} \cdot \lambda_{Mt} \Rightarrow M_{t+1} = a_t - b_t \cdot R^M_{t+1} \Leftrightarrow R^M_{t+1} \text{ SAT. MIN } \sigma^2_t(R_{t+1}) \text{ S.T. } E_t(R_{t+1}) = E$$

$$0 = E_t(M_{t+1} \cdot R^{ei}_{t+1})$$

COUNTER EXAMPLE: CONSUMPTION, AN "UNCONDITIONAL MODEL"

$$M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \text{ } \underline{\text{NOT}} \text{ } \beta_t, \gamma_t \quad \text{SAT. } 0 = E_t(M_{t+1} R^{ei}_{t+1}) \Leftrightarrow E(M_{t+1} R^e_{t+1} z_t)$$

PROBLEM

$$0 = E_t\Big[(\underbrace{a_t - b_t R^M_{t+1}}_{M_{t+1}}) R^{ei}_{t+1}\Big] \not\Rightarrow 0 = E\Big[(a - b R^M_{t+1}) R^{ei}_{t+1}\Big] \quad \text{PROOF TRY IT!}$$

- A CONDITIONAL MODEL DOES NOT IMPLY AN UNCONDITIONAL MODEL. R^{MV}_{t+1} ON CONDITIONAL MVF $\not\Rightarrow$ ON UNCONDITIONAL MVF
- AN UNCONDITIONAL MODEL (WHICH PRICES ALL MGD PORTFOLIOS) $\Rightarrow$ A C. MODEL! R^{MV}_{t+1} ON $\underline{\text{UC}}$ MVF $\Rightarrow$ ON C. MVF [INCLUDING MGD PORTFOLIOS]

• SOLUTION (PARTIAL)

MODEL CONDITIONING INFO.

"MGD"

$$M_t = a(z_t) - b(z_t) R^M_{t+1} = a_0 + a_1 z_t - b_0 R^M_{t+1} - b_1 z_t R^M_{t+1}$$

A CONDITIONAL CAPM + ONE INFO VARIABLE z_t = A 3 FACTOR UNCONDITIONAL MODEL.

• PARTIAL SOLUTION

⇒ 1) CONDITIONING INFO: AGENT, VARIABLES, MEAN

2) GOOD MODELS DO NOT ASSUME AGENTS ONLY SEE OUR VARIABLES

3) TS = CS MANAGED PORTFOLIOS