13.3 Term structure models

13.3.1 Expectations hypothesis model

- Simplest "model" a) short rate b) expectations to get other prices

$$y_{t+1}^{(1)} - \delta = \phi(y_t^{(1)} - \delta) + \varepsilon_{t+1}$$
$$f_t^{(2)} = E_t(y_{t+1}^{(1)}) = \delta + \phi(y_t^{(1)} - \delta)$$
$$f_t^{(3)} = E_t(y_{t+2}^{(1)}) = \delta + \phi^2(y_t^{(1)} - \delta)$$
$$f_t^{(N)} = E_t(y_{t+N-1}^{(1)}) = \delta + \phi^{N-1}(y_t^{(1)} - \delta)$$

$$\begin{aligned}
y_t^{(2)} &= \frac{1}{2}\left[E_t(y_{t+1}^{(1)}) + y_t^{(1)}\right] \\
&= \frac{1}{2}\left[\delta + \phi(y_t^{(1)} - \delta) + y_t^{(1)}\right] \\
y_t^{(2)} - \delta &= \frac{1+\phi}{2}\left(y_t^{(1)} - \delta\right) \\
&\dots \\
y_t^{(N)} - \delta &= \frac{1+\phi+\phi^2+..}{N}\left(y_t^{(1)} - \delta\right) = \frac{1-\phi^N}{N(1-\phi)}\left(y_t^{(1)} - \delta\right)
\end{aligned}$$

Result:

1 Different shapes, upward and downward sloping. Up when short rates expected to rise.

More complex shapes? Move past AR(1)!

2. No average slope – $E(y^{(i)})$ all the same. Well, we imposed expectations!

3. Yields move in lockstep – a "1 factor model"

13.3.2 "Term structure model" in general

Ingredients:

1) Write a time series model for the discount factor in discrete or continuous time.

2a) Solve M forward, $M_{t,t+n} = M_{t+1}M_{t+2}...M_{t+n}$. Then

$$P_t^{(n)} = E_t\left[M_{t,t+n}\right]$$

2b) Solve differential / difference equation, i.e. solve $P_t^{(n)}$ "backward" from $P_t^{(0)} = 1$,

$$P_t^{(n)} = E_t\left[M_{t,t+1}P_{t+1}^{(n-1)}\right]$$

Result: $P_t^{(n)}$ = function of state variables that drive M.

This is easiest for logs. Translation from levels to logs: either continuous time or lognormal distributions.

13.3.3 Discrete-time single-factor Vasicek

Here: A standard "single factor model" – "discrete-time Vasicek." The end result:

$$\left(y_{t+1}^{(1)}-\delta\right)=\phi\left(y_t^{(1)}-\delta\right)+v_{t+1}.$$

$$\begin{aligned}
f_t^{(2)} &= \delta+\phi\left(y_t^{(1)}-\delta\right)-\left[\frac{1}{2}+\lambda\right]\sigma_\varepsilon^2 \\
f_t^{(3)} &= \delta+\phi^2\left(y_t^{(1)}-\delta\right)-\left[\frac{1}{2}(1+\phi)^2+\lambda(1+\phi)\right]\sigma_\varepsilon^2 \\
&\ldots.
\end{aligned}$$

$$\begin{aligned}
y_t^{(2)} &= \delta+\frac{(1+\phi)}{2}\left(y_t^{(1)}-\delta\right)-\frac{1}{2}\left(\frac{1}{2}+\lambda\right)\sigma_\varepsilon^2 \\
y_t^{(3)} &= \delta+\frac{(1+\phi+\phi^2)}{3}\left(y_t^{(1)}-\delta\right)-\frac{1}{3}\left\{\frac{1}{2}\left[1+(1+\phi)^2\right]+\lambda\left[1+(1+\phi)\right]\right\}\sigma_\varepsilon^2 \\
&\ldots.
\end{aligned}$$

Intuition for now: the first equation tells you where interest rates are going over *time*. The second and third sets of equations tell you where each forward rate is at any date, depending only on where the short rate is on that date; a single factor model.

Derivation:

- Suppose m follows the time series model.

$$\begin{aligned}
x_{t+1}-\delta &= \phi\left(x_t-\delta\right)+\varepsilon_{t+1} \\
m_{t+1} &= \ln M_{t+1}=-\frac{1}{2}\lambda^2\sigma_\varepsilon^2-x_t-\lambda\varepsilon_{t+1}
\end{aligned}$$

  This is just a model for m, with a convenient "state variable" x_t. Intuition: x_t shifts the *mean* of $\ln m_{t+1}$ around. If you remember that $R^f=1/E(m)$, you can see that specifying a model for the mean of m is the key to thinking about interest rates. The $1/2\sigma^2$ term just shifts the mean of $\ln m_t$ down, and offsets a $1/2\sigma^2$ term which will pop up later. To be specific, ε and v are iid Normal with $\sigma_\varepsilon^2, \sigma_v^2, \sigma_{\varepsilon v}$. δ, ρ, λ are free parameters; we'll pick these to make the model fit as well as possible.

- Bond prices.

$$P_t^{(n)}=E_t\left(M_{t+1}M_{t+2}\ldots.M_{t+n}\right)$$

  This is easier to do recursively,

$$\begin{aligned}
P_t^{(0)} &= 1 \\
P_t^{(n)} &= E_t\left(m_{t+1}P_{t+1}^{(n-1)}\right).
\end{aligned}$$

- Here we go.

$$P_t^{(1)}=E_t\left(M_{t+1}\right)=E_te^{m_{t+1}}$$

$$P_t^{(1)}=e^{-\frac{1}{2}\lambda^2\sigma_\varepsilon^2-x_t+\frac{1}{2}\lambda^2\sigma_\varepsilon^2}=e^{-x_t}$$

$$\begin{aligned} p_t^{(1)} &= \ln E_t\left(m_{t+1}\right) = -x_t \\ y_t^{(1)} &= x_t \end{aligned}$$

Now you see why I set up the problem with the $1/2\lambda^2\sigma_\varepsilon^2$ to begin with! *The one year interest rate "reveals the latent state variable x_t"*

- I could have written the model as

$$\begin{aligned} y_{t+1}^{(1)} - \delta &= \phi\left(y_t^{(1)} - \delta\right) + \varepsilon_{t+1} \\ m_{t+1} &= -\frac{1}{2}\lambda^2\sigma_\varepsilon^2 - y_t^{(1)} - \lambda\varepsilon_{t+1} \end{aligned}$$

"a short rate process plus a market price of risk." Then, by taking $-\ln E_t\left(M_{t+1}\right)$ I would have checked that the $y_t^{(1)}$ the model *produces* is the same $y_t^{(1)}$ I *started* with. Take your pick. Which is more confusing: a) starting with an x_t you "can't see" and then showing that it turns out to be $y_t^{(1)}$? b) starting with an assumed $y_t^{(1)}$ process and then showing that it's in fact the one period rate, that the model is "self-consistent" (in the language of CP appendix.)

- On to the next price.

$$\begin{aligned} P_t^{(2)} &= E_t\left(m_{t+1}P_{t+1}^{(1)}\right) = E_t\left(e^{m_{t+1}+p_{t+1}^{(1)}}\right) \\ &= E_t\left(e^{-\frac{1}{2}\lambda^2\sigma_\varepsilon^2 - x_t - \lambda\varepsilon_{t+1} - x_{t+1}}\right) \end{aligned}$$

$$P_t^{(2)} = E_t\left(e^{-\frac{1}{2}\lambda^2\sigma_\varepsilon^2 - x_t - \lambda\varepsilon_{t+1} - \delta - \phi(x_t-\delta) - \varepsilon_{t+1}}\right)$$

$$P_t^{(2)} = E_t\left(e^{-2\delta - (1+\phi)(x_t-\delta) - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 - (1+\lambda)\varepsilon_{t+1}}\right)$$

$$\begin{aligned} p_t^{(2)} &= -2\delta - (1+\phi)\left(x_t-\delta\right) - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 + \frac{1}{2}(1+\lambda)^2\sigma_\varepsilon^2 \\ p_t^{(2)} &= -2\delta - (1+\phi)\left(x_t-\delta\right) + (\frac{1}{2}+\lambda)\sigma_\varepsilon^2 \end{aligned}$$

- From prices, we find yields and forwards,

$$y_t^{(2)} = \delta + \frac{(1+\phi)}{2}\left(x_t-\delta\right) - \frac{1}{2}\left(\frac{1}{2}+\lambda\right)\sigma_\varepsilon^2$$

$$\begin{aligned} f_t^{(2)} &= p_t^{(1)} - p_t^{(2)} \\ &= -\delta - \left(x_t-\delta\right) + 2\delta + (1+\phi)\left(x_t-\delta\right) - (\frac{1}{2}+\lambda)\sigma_\varepsilon^2 \\ &= \delta + \phi\left(x_t-\delta\right) - (\frac{1}{2}+\lambda)\sigma_\varepsilon^2 \end{aligned}$$

- Now the rest of the maturities. You can "solve the discount rate forward and integrate"

$$\begin{aligned} p_t^{(3)} &= \log E_t\left(M_{t+1}M_{t+2}M_{t+3}\right) \\ &= \log E_t e^{-\frac{1}{2}\lambda^2\sigma_\varepsilon^2 - x_t - \lambda\varepsilon_{t+1} - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 - x_{t+1} - \lambda\varepsilon_{t+2} - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 - x_{t+2} - \lambda\varepsilon_{t+3}} \\ &= \log E_t e^{-3\delta - \frac{3}{2}\lambda^2\sigma_\varepsilon^2 - (1+\phi+\phi^2)(x_t-\delta) - \lambda\varepsilon_{t+1} - \lambda\varepsilon_{t+2} - \lambda\varepsilon_{t+3} - (1+\phi)\varepsilon_{t+1} - \varepsilon_{t+2}} \end{aligned}$$

This will work after much algebra

- Instead, let's do it recursively "derive a differential equation for price as a function of state variables." Guess

$$P_t^{(n)} = A_n - B_n(x_t - \delta)$$

then

$$\begin{aligned}
P_t^{(n)} &= E_t\left(M_{t+1}P_{t+1}^{(n-1)}\right)\\
A_n - B_n(x_t-\delta) &= \log E_t\left(\exp\left(-\frac{1}{2}\lambda^2\sigma_\varepsilon^2 - x_t - \lambda\varepsilon_{t+1}\right)\exp\left(A_{n-1} - B_{n-1}(x_{t+1}-\delta)\right)\right)\\
&= \log E_t\left(\exp\left(-\frac{1}{2}\lambda^2\sigma_\varepsilon^2 - x_t - \lambda\varepsilon_{t+1} + A_{n-1} - B_{n-1}\phi(x_t-\delta) - B_{n-1}\varepsilon_{t+1}\right)\right)\\
&= \log E_t\left(\exp\left(-\delta + A_{n-1} - (1+B_{n-1}\phi)(x_t-\delta) - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 - \lambda\varepsilon_{t+1} - B_{n-1}\varepsilon_{t+1}\right)\right)\\
A_n - B_n(x_t-\delta) &= -\delta + A_{n-1} - (1+B_{n-1}\phi)(x_t-\delta) + \left(B_{n-1}\lambda + \frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2
\end{aligned}$$

The constant and the term multiplying x_t must separately be equal. Thus,

$$B_n = 1 + B_{n-1}\phi$$

$$A_n = -\delta + A_{n-1} + \left(B_{n-1}\lambda + \frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2$$

We have "transformed the solution of a stochastic differential equation plus integral to the solution of an ordinary differential equqation.

- That's easy to solve

$$\begin{aligned}
B_0 &= 0\\
B_1 &= 1\\
B_2 &= 1+\phi\\
B_3 &= 1+\phi+\phi^2\\
B_n &= \sum_{j=0}^{n-1}\phi^j = \frac{1-\phi^n}{1-\phi}
\end{aligned}$$

$$\begin{aligned}
A_0 &= 0\\
A_1 &= -\delta\\
A_2 &= -2\delta + \left(\lambda + \frac{1}{2}\right)\sigma_\varepsilon^2\\
A_3 &= -3\delta + \left[(1+\phi)\lambda + \frac{1}{2}(1+\phi)^2\right]\sigma_\varepsilon^2\\
A_4 &= -4\delta + \left[\left(1+\phi+\phi^2\right)\lambda + \frac{1}{2}\left(1+\phi+\phi^2\right)^2\right]\sigma_\varepsilon^2
\end{aligned}$$

You see the pattern from here

- Yields, forwards, returns, etc. follow. $y_t^{(n)} = -1/n \times p_t^{(n)}$. Forwards are even simpler,

$$\begin{aligned}
f_t^{(n)} &= p_t^{(n-1)} - p_t^{(n)}\\
&= (A_{n-1} - A_n) - (B_{n-1} - B_n)(x_t - \delta)\\
&= -\delta + \left(B_{n-1}\lambda + \frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2 + \phi^{n-1}(x_t-\delta)
\end{aligned}$$

- Result:
$$\left(y_{t+1}^{(1)}-\delta\right)=\phi\left(y_t^{(1)}-\delta\right)+\varepsilon_{t+1}.$$

$$\begin{aligned}
f_t^{(2)} &= \delta+\phi\left(y_t^{(1)}-\delta\right)-(\frac{1}{2}+\lambda)\sigma_\varepsilon^2 \\
f_t^{(3)} &= \delta+\phi^2\left(y_t^{(1)}-\delta\right)-\left[\frac{1}{2}(1+\phi)^2+\lambda(1+\phi)\right]\sigma_\varepsilon^2 \\
f_t^{(4)} &= \delta+\phi^3\left(y_t^{(1)}-\delta\right)-\left[\frac{1}{2}\left(1+\phi+\phi^2\right)^2+\lambda(1+\phi+\phi^2)\right]\sigma_\varepsilon^2 \\
& \dots
\end{aligned}$$

$$\begin{aligned}
y_t^{(2)} &= \delta+\frac{(1+\phi)}{2}\left(y_t^{(1)}-\delta\right)-\frac{1}{2}\left(\frac{1}{2}+\lambda\right)\sigma_\varepsilon^2 \\
y_t^{(3)} &= \delta+\frac{(1+\phi+\phi^2)}{3}\left(y_t^{(1)}-\delta\right)-\frac{1}{3}\left\{\frac{1}{2}\left[1+(1+\phi)^2\right]+\lambda\left[1+(1+\phi)\right]\right\}\sigma_\varepsilon^2 \\
& \dots
\end{aligned}$$

1. Just like EH but now a risk premium!
2. Shapes: A steady decline from σ^2 terms, (risk premim) + exponential decay from $y^{(1)}-E(y^{(1)})$. (expectations hypothesis)
3. "Short rate process" plus "one factor model." All yields move in lockstep indexed by $y_t^{(1)}$ (or any other yield). The shape is tied to the level. It looks like "you can price other bonds by arbitrage" but that is only because we restrict our model to have one factor.
4. $y^{(1)}$ is also sufficient to *forecast* all yields.
5. Risk premium comes from $cov(m,y^{(1)})=\lambda\sigma_\varepsilon^2$ "market price of interest rate risk". If there were a security whose payoff were ε_{t+1} its price would be driven by $cov(m,\varepsilon_{t+1})$.
6. The premium can go either way depending on the sign of λ. My guess: lower $y_{t+1}^{(1)}$means higher m (bad state) means + ($m=..-\varepsilon$) sign and negative premium. This is a typical result. The real term structure ought to slope down, as long term bonds are safer for long-term investors. However, as long as we separate market prices of risk from consumption and interest data (as we did with the CAPM!) we can incorporate an upward sloping yield curve with $\lambda<0$
7. The risk premium is constant over time though – as we'll see not in data.
8. "Risk neutrality" $\lambda=0$ does not mean "expectations" since there is another term. This is Another force for typical downward slope. However, it's quantitatively very small, since $\sigma_\varepsilon\approx 0.01$
9. Another way to see risk premia is to look at returns,
$$\begin{aligned}
r_{t+1}^{(n)} &= p_{t+1}^{(n-1)}-p_t^{(n)}=(A_{n-1}-A_n)-B_{n-1}(x_{t+1}-\delta)+B_n(x_t-\delta) \\
&= \delta-\left(B_{n-1}\lambda+\frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2-B_{n-1}(x_{t+1}-\delta)+B_n(x_t-\delta)
\end{aligned}$$

Expected returns

$$
\begin{aligned}
E_t r_{t+1}^{(n)} &= \delta - \left(B_{n-1}\lambda + \frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2 + (B_n - B_{n-1}\phi)(x_t - \delta) \\
E_t r_{t+1}^{(n)} &= \delta - \left(B_{n-1}\lambda + \frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2 + (x_t - \delta) \\
E_t r x_{t+1}^{(n)} &= -\left(B_{n-1}\lambda + \frac{1}{2}B_{n-1}^2\right)\sigma_\varepsilon^2
\end{aligned}
$$

You see the expected returns differ by maturity, but the risk premium is constant over time – not what Fama-Bliss find.

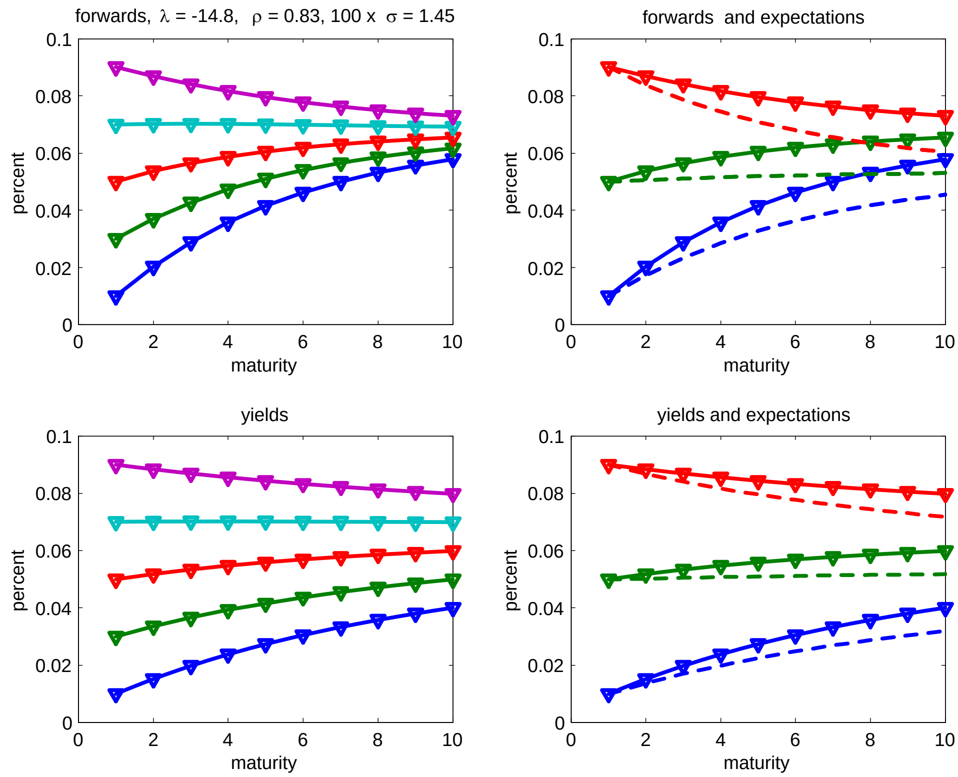
10. The limiting yield and forward rate are constants. There is no true "level" shift. We'll see this is quite general – "level shifts" imply an arbitrage opportunity at the long end of the yield curve.
11. Yields can be negative – they are normally distributed here. $M > 0$ means $P > 0$ not $P < 1$. The CIR model fixes this up.

- Let's see an example.

1. I chose some parameters to fit the FB zero coupon bond data. I ran a regression of $y_{t+1}^{(1)}$ on $y_t^{(1)}$ to get ρ; I took the variance of errors from that regression to get σ_ε; I took the mean $\delta = E\left(y_t^{(1)}\right)$. Finally, I picked the market price of risk λ to fit the average 5 year forward spread:

$$
\begin{aligned}
f_t^{(5)} &= \delta + \rho^4\left(y_t^{(1)} - \delta\right) - \left[\frac{1}{2}\left(1+\rho+\rho^2+\rho^3\right)^2 + \lambda(1+\rho+\rho^2+\rho^3)\right]\sigma_\varepsilon^2 \\
\lambda &= -\frac{E\left(f_t^{(5)}\right) - \delta}{(1+\rho+\rho^2+\rho^3)\sigma_\varepsilon^2} - \frac{1}{2}\left(1+\rho+\rho^2+\rho^3\right)
\end{aligned}
$$

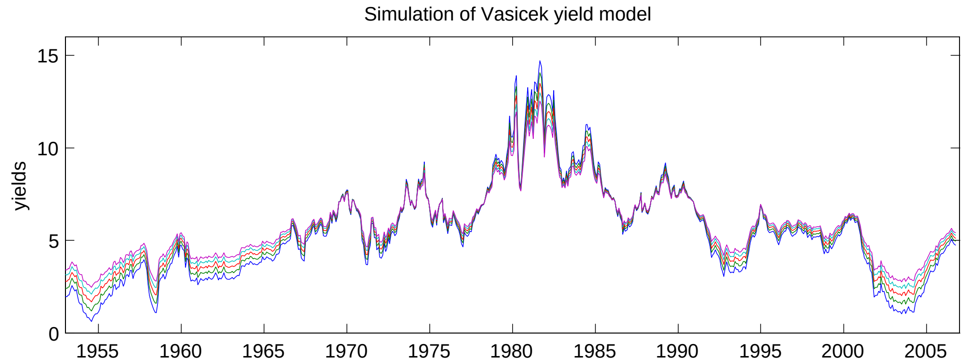
2. I plot $y_t^{(n)}$ for a bunch of $y_t^{(1)}$. The dashed lines in the right hand graph give the expectations hypothesis terms from above, so you can see the distortion from risk aversion λ and the Jensen's inequatlity σ_ε^2 term.

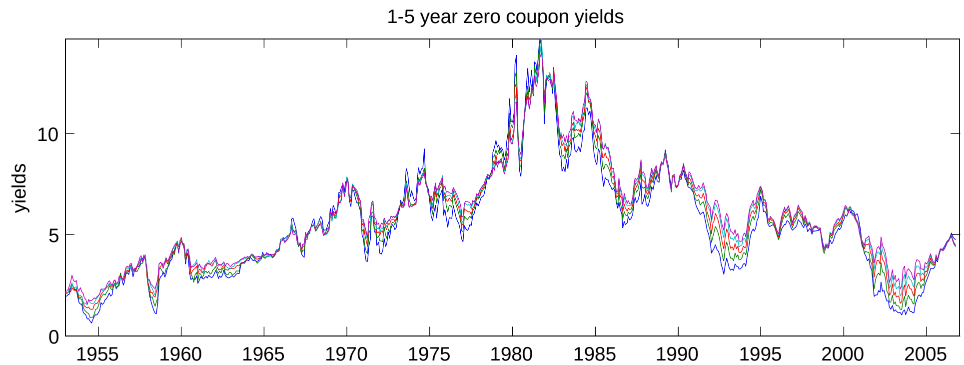


Cool! This captures some basic patterns; yields are upward sloping when lower, downward sloping when higher. The substantial risk premium I estimated to match the average upward slope does introduce a substantial deviation of the model from expectations at the long end.

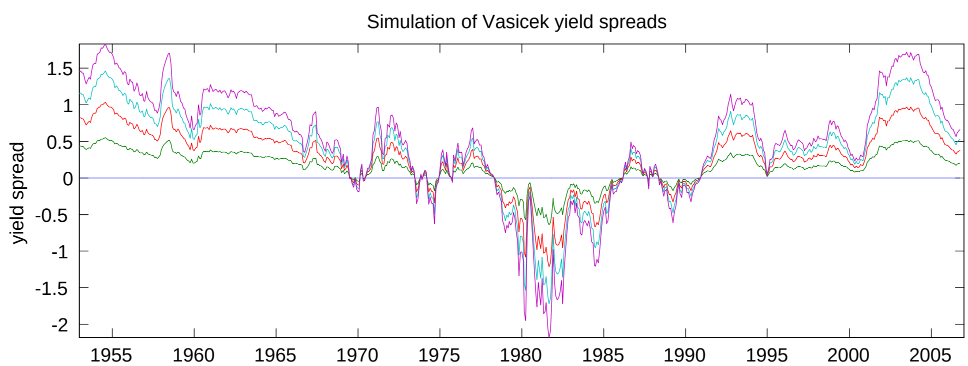
3. Note already: the parameters ϕ, λ can be chosen to match the *cross section* of yields – the shapes of these curves – or the *time series* – the AR(1) coefficient of the short rate and the expected bond returns. These do not necessarily give the same answer, a sign of model misspecification.

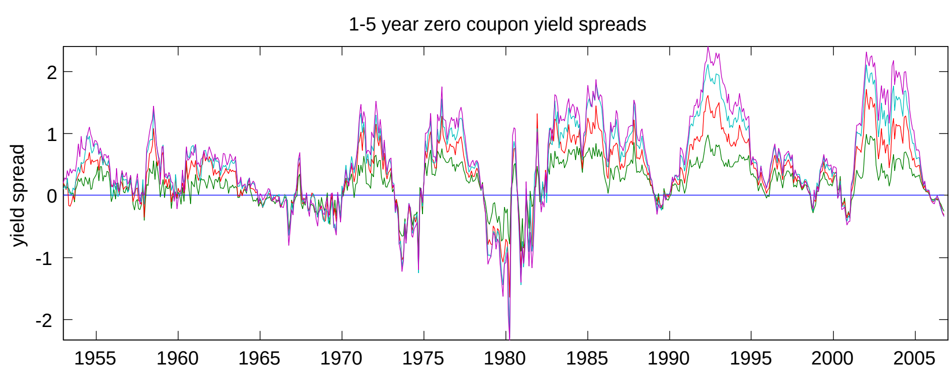
4. Take the history of $y_t^{(1)}$. Find the model-implied $y_t^{(n)}$: compare with data.





5. You can see a decent fit – upwward sloping yields when the interest rate is low. But you can see yields are going up to a constant long-term value, rather than some sort of "local mean. This is clearer if we plot spreads,





Answer: we need a two-factor model....