## Brandt, Cochrane, Santaclara

Three puzzles in international finance pretty much define the field.

1. UIP (Just studied): $r^f - r^d$ seems to imply appreciation, not depreciation, at least for a while, and corresponding profits.

2. The volatility of exchange rates.

$$\sigma\left(\ln\frac{e_{t+1}}{e_t}\right) = 15\%$$

   This is not matched 1-1 by inflation, so real exchange rates vary a lot. (Mussa) Real relative prices across borders change when exchange rates change, suggesting "sticky" nominal prices. When countries move from floating to fixed, relative prices across countries (sausage in Munich/Pizza in rome) become more stable, and relative prices of tradeables/nontradeables (Pizza in Rome/Oil in Rome) become more stable. Put bluntly, why did this happen?


3. Savings = investment and poor risk sharing across countries. One is about allocation across time, the other about allocation across states.

   (a) Permanent income logic means that temporary high Y should be exported and then returned later. Also good news about future output (China opening) should lead to a consumption boom and huge imports of capital. Instead, China finance investment from domestic savings and exported the whole time. (This is an open economy facing world interest rates. If the whole world sees a boom, interest rates rise.)

   (b) Complete markets, Pareto-Optimum means

$$\max E\left[\lambda_1 \sum \beta^t u(c_{1t}) + \lambda_2 \sum \beta^t u(c_{2t})\right] \text{ s.t. } c_{1t} + c_{2t} = C_t$$

$$
\begin{aligned}
\lambda_1 \beta^t u'(c_{1t}) &= \delta_t \\
\lambda_2 \beta^t u'(c_{2t}) &= \delta_t \\
\lambda_1 \beta^t u'(c_{1t}) &= \lambda_2 \beta^t u'(c_{2t}) \\
\beta \frac{u'(c_{1t})}{u'(c_{1t-1})} &= \beta \frac{u'(c_{2t})}{u'(c_{2t-1})} \\
\left( \frac{c_{1t}}{c_{1t-1}} \right)^{-\gamma} &= \left( \frac{c_{2t}}{c_{2t-1}} \right)^{-\gamma}
\end{aligned}
$$

In fact, $cor(\Delta c_i, \Delta c_j)$ is small (numbers follow). Worse, consumption correlations are less than output correlations.

(c) In both cases, I find it hilarous that as the world starts to look more like our models, people think this is a problem. "Global imbalances" is the buzzword for the idea that we need to *slow down* trade surpluses and deficits. Mortgage backed securities did a great job of sharing risk around the world.

4. "Home bias" in portfolios. US people hold mostly US equities, UK people hold more UK equities and so forth. This is only a puzzle however relative to a world capm model, in which the investor has no job, cares equally about consumption from all countries, etc. There are lots of easy reasons it's optimal for portfolios to focus on your own country, as in my Earth vs. Mars example.

5. International is RIPE for work, as witnessed by lustig et al, tarek hassan. Simple models are making big progress.

**This paper:**

We can connect domestic and foreign discount factors by a simple change of units.

$$
\begin{aligned}
M^f_{t+1} &= M^d_{t+1} \frac{S_{t+1}}{S_t} \\
\frac{utils_{t+1}}{Euro_{t+1}} &= \frac{utils_{t+1}}{\$_{t+1}} \frac{\$_{t+1}/E_{t+1}}{\$_t/E_t} \\
m^f_{t+1} &= m^d_{t+1} + \ln \frac{S_{t+1}}{S_t}
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
1 &= E(M^d_{t+1} R_{t+1}) \\
&= E(M^d_{t+1} \frac{S_{t+1}}{S_t} \frac{S_t}{S_{t+1}} R_{t+1}) \\
&= E(M^f_{t+1} R^f_{t+1})
\end{aligned}
$$

where $R^f =$ any return (domestic or foreign) expressed in foreign currency.

This is cool! Exchange rates let you see mrs, directly, ex post! Well, they let you see *differences* in mrs, but that's something.

Important – distinguish "discount factor for returns expressed in domestic currency" from "discount factor that only prices domestic returns." The latter makes no sense unless segmented market of some sort. *All* we're doing here is saying that we can find a discount factor that predicts a *given* set of returns converted to Euros from a discount factor that prices *the same* returns expressed in dollars. (Which is, when you see it, rather trivial.) We are *not* constructing a discount factor that prices Euro stocks from a discount factor that (only) prices dollar stocks.

You can do the same thing with real vs. nominal discount factors. Just mulitply and divide by $\pi$.

Now,

$$\begin{aligned} \ln \frac{S_{t+1}}{S_t} &= m_{t+1}^f - m_{t+1}^d \\ \sigma^2 \left(s_{t+1} - s_t\right) &= \sigma^2 \left(m_{t+1}^f\right) + \sigma^2 \left(m_{t+1}^d\right) - 2\rho\sigma(m)\sigma(m) \end{aligned}$$

What does it take to fit the facts?

$$\sigma\left(\Delta s_{t+1}\right) = 15\%$$

1. Asset pricing, "risk sharing is better than you think." From asset markets, $\frac{E(R^e)}{\sigma(R^e)} <\approx \sigma(m)$ we need at least $\sigma(m) = 50\%$. Since $\sigma(m)$ is much bigger than $\sigma(e)$ we need a lot of positive corrleation.

$$\begin{aligned} 0.15^2 &= 2 \times 0.50^2 - 2 \times \rho \times 0.50^2 \\ 0.0225 &= 0.50\left(1-\rho\right) \\ 0.045 &= \left(1-\rho\right) \\ \rho &= 0.955 \end{aligned}$$

   "Risk sharing is better than you think" meaning marginal utility growth is very correlated across countires.

2. Asset pricing, "or exchange rates are too smooth." Imposing $\rho = 0$,

$$\sigma(\Delta s_{t+1}) = \sqrt{2}\sigma(m) = 1.41 \times 0.5 = 0.71$$

   We should see 70% variation in exchange rates!

3. Consumption. If we use consumption data, $\Delta c$, small risk aversion$\gamma$, and $\rho = 0$ as suggested by the data, no matter what we do with $\rho$ on the right hand side $\sigma(m)$ is just not enough to add up to the observed $\sigma(\Delta s)$. $\sigma(m) = \gamma\sigma(\Delta c)$.

$$\begin{aligned} 0.15^2 &= 2 \times \gamma \times \sigma^2(\Delta c) - 2\rho \times \gamma \times \sigma^2(\Delta c) \\ &= 2 \times \gamma \times \sigma^2(\Delta c) \times (1-\rho) \\ \frac{0.15}{\sqrt{2}} &= 0.106 = \gamma\sigma(\Delta c)(1-\rho) \end{aligned}$$

This isn't as bad as the equity premium (that's the whole point), where we needed $\gamma\sigma(\Delta c) = 0.5$. It's still not easy. With $\rho = 0$ and $\sigma(\Delta c) = 0.01$ we need $\gamma = 10$. $\sigma(\Delta c) = 2\%$ gets us down to $\gamma = 5$. And surely there's some positive correlation. In sum, you can see that models (especially models with risk sharing) have trouble producing *enough* exchage rate movement.

**The paper:** computes a "risk sharing index"

$$1 - \frac{\sigma^2(\ln m^f - \ln m^d)}{\sigma^2(\ln m^f) + \sigma^2(m^d)} = 1 - \frac{\sigma^2(\ln e_{t+1}/e_t)}{\sigma^2(\ln m^f) + \sigma^2(m^d)}$$

Why? $\ln m^f = 2\ln m^d$ also violates risk sharing, but *correlation* is one.

Procedure: Just like Hansen-Jagannathan. Find the minimum variance discount factors $m$ to price both domestic and foreign assets, expressed in dollars, and vice versa. We use continuous time so we can do logs vs. levels ("E(log) = log(E) theorem" is true in continuous time, with $1/2\sigma^2$ terms)

Continuous time

$$\begin{aligned} \Lambda^d &= e\Lambda^f \\ d\ln\Lambda^d &= d\ln e + d\ln\Lambda^f \end{aligned}$$

$$\begin{aligned} \frac{dS}{S} &= (r+\mu)\,dt - \sigma dB \\ \frac{d\Lambda}{\Lambda} &= -rdt - \mu'\Sigma^{-1}\sigma dB \\ \frac{d\ln\Lambda}{\Lambda} &= -\left[r + \frac{1}{2}\mu'\Sigma^{-1}\mu\right]dt - \mu'\Sigma^{-1}\sigma dB \\ \sigma^2\left(\frac{d\ln\Lambda}{\Lambda}\right) &= \mu'\Sigma^{-1}\mu \end{aligned}$$

so the regular calculation works in logs in continuous time. **Table 2,3** gives the basic calculation.

**Strongly recommended reading:**

The introduction on transport costs (Earth vs. Mars) and incomplete markets, and "reconciliation" p. 692 i.e. apples and oranges.

*Earth vs. Mars*: Suppose there are complete financial assets and communication but no goods may flow. If mars gets a good shock, mars stock goes up. The exchange rate must go down. $m^f$ and $m^d$ must be uncorrelated in the end. Knowing this, there is no advantage to Mars stock in the first place, so your portfolio should be completely home biased. *Complete financial assets do not imply "perfect risk sharing" nor constant exchange rates, nor absence of home bias.* Money, capital can't "flow". International is about transport costs, not markets.

*Incomplete markets.* Now $m^i = m^* + \varepsilon^i$ again. It is *not* true that $m^d = m^f + \Delta s$ for arbitrary $m$ pairs. It *is* true that for any $m^d$ that prices assets expressed in dollars, we can construct *an* $m^f$ that prices assets expressed in Euros. Again, this is just a change of units. But that discount factor may not equal foreign consumption growth. It *is* true that $m^{*d} = m^{*f} + \Delta s$. Thus *minimum-variance discount factors in the payoff space do obey the identity.* In this sense, what we are learning is "do transport costs mean that we are not able to use asset markets to share as many risks as possible?"

The paper also asks if incomplete markets are quantitatively plausible. $\sigma(m)$ rises as $m$ becomes less correlated.

How much international finance uses vs. does not use complete markets?? Be very careful here. if

$$e = m^d - m^f$$

and then

$$e = \gamma \Delta c^d_{t+1} - \gamma \Delta c^f_{t+1}$$

you *are* assuming complete markets. If

$$e = proj(\Delta c | assets) - proj(\Delta c^f | assets)$$

then you're not.

**Misconceptions:**

1. *Each investor is allowed to invest in all assets* – the HJ, minimum variance discount factor for all assets as viewed by each investor. Our equation only applies as a change of units. These are NOT the minimum variance discount factor for domestic assets and the minimum discount factor for foreign assets. Why not? You can compute such quantities, but they are not connected by the exchange rate.

   The paper does *nothing* about "what if there are asset market frictions so you can't trade each other's assets?" We allow markets to be incomplete, but once D can buy them, so can F. Would it be interesting to give the countries fundamentally different spaces, or (the same thing) prices that are different by shadow costs or transactions costs as well as by exchange rates? Yes, but we didn't do it.

2. The "discount factor" is *not* the "optimal portfolio" the "market portfolio", or a portfolio anyone actually holds. The *correlation* of discount factors means *nothing* about correlation of portfolios. For example, if our income shocks are the same, we can have very correlated discount factors, very correlated $m^*$ but utterly different portfolios.

3. The correlation of stock markets, which underlie the usual "benefits of international diversification," is not really in the calculation at all. (Technically, it is reflected in the $\Sigma$ part of $\mu'\Sigma^{-1}\mu$, but higher asset correlation *does not imply* higher discount factor correlation.) Again, we allow trade in both assets by both investors.

4. Converesly, the discount factor is not a portfolio that anyone holds, so highly correlated discount factors do not mean portfolios are highly correlated. Thus, also, home bias does not contradict our calculations.

# Habits

**Basic idea:**

- Consumption and habit picture.
- *Slow moving* is key. Note much macro is now using one period habit, a bad idea.
- One, business cycle related time varing risk aversion unites a lot of behavior we've studied. dp forecasts, price volatility, etc.
- Now!
- Is "habit" the mechanism? "leverage" or "irreversible durable goods" behaves the same way
- Also a laboratory for thinking about issues. Such as macro linearizations, conditional vs unconditional models, etc.
- A lot easier than long run risks, EZ, etc.
- Proud reverse-engineering. what must model be to produce the world we see? No difference between functional form and numbers!

**Model**

$$E\sum_{t=0}^{\infty}\delta^t\frac{(C_t-X_t)^{1-\gamma}-1}{1-\gamma}. \tag{10}$$

$$S_t\equiv\frac{C_t-X_t}{C_t}.$$

More s is good times. Less s is bad times.

$$\eta_t\equiv-\frac{C_t u_{cc}(C_t,X_t)}{u_c(C_t,X_t)}=\frac{\gamma C(C-X)^{-\gamma-1}}{(C-X)^{-\gamma}}=\frac{\gamma C}{(C-X)}=\frac{\gamma}{(\frac{C-X}{C})}=\frac{\gamma}{S_t}.$$

Not risk aversion! $rra=V_{WW}/(WvW)$ tells you bets on wealth. $W\neq C$ in general. (more later)

How does consumption adapt to habit? Like $X_t=\sum\phi^j C_{t-j}=\phi X_{t-1}+C_t$

$$s_{t+1}=(1-\phi)\bar{s}+\phi s_t+\lambda(s_t)(c_{t+1}-c_t-g). \tag{11}$$

$\phi$, $g$ and $\bar{s}$ are parameters. $\lambda(s_t)$ the *sensitivity function.*

a) use log s to keep S always positive!

b) $\lambda$ allows chs in m, which we know we need. (s is state variable, m(s)). Reverse engineer it below. (what must $\lambda$ be to produce the world we see)

Technology

$$\Delta c_{t+1} = g + v_{t+1}; \quad v_{t+1} \sim i.i.d.\ \mathcal{N}(0, \sigma^2). \tag{12}$$

**Marginal utility**

Habit is external, marginal utility is

$$u_c(C_t, X_t) = (C_t - X_t)^{-\gamma} = S_t^{-\gamma} C_t^{-\gamma}.$$

If *internal,* forward looking terms, $u_x(t+j)\partial X_{t+j}/\partial C_t$. In the end add nothing but complication (later) Convenience.

$$M_{t+1} \equiv \delta \frac{u_c\left(C_{t+1}, X_{t+1}\right)}{u_c\left(C_t, X_t\right)} = \delta \left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma}.$$

$$\begin{aligned} M_{t+1} &= \delta G^{-\gamma} e^{-\gamma(s_{t+1} - s_t + v_{t+1})} \\ s_{t+1} - s_t &= (1-\phi)\left(s_t - \bar{s}\right) + \lambda\left(s_t\right) v_{t+1}. \\ M_{t+1} &= \delta G^{-\gamma} e^{-\gamma[(\phi-1)(s_t - \bar{s}) + (1+\lambda(s_t))v_{t+1}]}. \end{aligned}$$

Note "amplificatin" for one period, a shock to c moves S as well. Thus m moves more. 1 is the consumption, $\lambda$ is the s movment.

**HJ Bound**

$$max \frac{E_t(R^e_{t+1})}{\sigma_t(R^e_{t+1})} = \frac{\sigma_t(M_{t+1})}{E_t(M_{t+1})}$$

$$M = e^m; \frac{\sigma(M)}{E(M)} = \frac{\sqrt{E(M^2) - E(M)^2}}{E(M)} = \frac{\sqrt{e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}}}{e^{\mu + \sigma^2/2}} = \sqrt{e^{\sigma^2} - 1}$$

$$\frac{E_t(R^e_{t+1})}{\sigma_t(R^e_{t+1})} = \frac{\sigma_t(M_{t+1})}{E_t(M_{t+1})} = \left(e^{\gamma^2\sigma^2[1+\lambda(s_t)]^2} - 1\right)^{\frac{1}{2}} \approx \gamma\sigma[1 + \lambda(s_t)].$$

Remind you : data say time-varying sharpe ratio, and we want high sharpe in bad times. Thus $\lambda$ should be declining in s

**Risk free rate**

$$R^f_t = 1/E_t\left(M_{t+1}\right).$$

$$M_{t+1} = \delta e^{-\gamma g} e^{-\gamma[(\phi-1)(s_t - \bar{s}) + (1+\lambda(s_t))v_{t+1}]}$$

$$r^f_t = -\ln(\delta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma^2\sigma^2}{2}\left[1 + \lambda(s_t)\right]^2.$$

- Intertemporal substitution vs. precautionary saving. Now both time-varying. Low S – desperate to borrow, but also worried about further declines. (now!)

- Choose $\lambda$ declining in $S$, and they can offset
- Paper: for rhetorical purposes a constant risk free rate. (Now regret, since people don't read the paragraph that shows to to have a time-varying risk free rate and fb regressions) This already means the square root-1 formula.

$$\begin{aligned} -r^f - \ln(\delta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) &= \frac{\gamma^2\sigma^2}{2}\left[1+\lambda(s_t)\right]^2 . \\ \frac{\sqrt{2}}{\gamma\sigma}\sqrt{\gamma g - r^f - \ln(\delta) - \gamma(1-\phi)(s_t - \bar{s})} - 1 &= \lambda(s_t) \end{aligned}$$

**Sensitivity function**

Some other pretty considerations, lead us to restrict $\bar{s}$ and the other parameters

$$\bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi}},$$

$$\lambda(s_t) = \frac{1}{\bar{S}}\left[\sqrt{1-2\left(s_t - \bar{s}\right)} - 1\right]$$

$$s_{\max} \equiv \bar{s} + \frac{1}{2}\left(1 - \bar{S}^2\right).$$

[Show or plot figure 1 of $\lambda$]

**Properties**

- Rf constant works,

$$\begin{aligned} r_t^f &= -\ln(\delta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma^2\sigma^2}{2}\left[1+\lambda(s_t)\right]^2 \\ &= -\ln(\delta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma^2\sigma^2}{2}\left[(1/\bar{S})\sqrt{1-2\left(s_t - \bar{s}\right)}\right]^2 \\ &= -\ln(\delta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma^2\sigma^2}{2}\left[\frac{1-\phi}{\sigma^2\gamma}\left(1-2\left(s_t - \bar{s}\right)\right)\right] \\ &= -\ln(\delta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) + \gamma\left(1-\phi\right)\left(s_t - \bar{s}\right) - \frac{\gamma}{2}\left(1-\phi\right) \\ r_t^f &= -\ln(\delta) + \gamma g - \frac{\gamma}{2}\left(1-\phi\right) = -\ln(\delta) + \gamma g - \left(\frac{\gamma}{\bar{S}}\right)^2\frac{\sigma^2}{2} \end{aligned}$$

- Note generalization

$$r_t^f = r_0^f - B\left(s_t - \bar{s}\right).$$

$$\bar{S} = \sigma\sqrt{\frac{\gamma}{1-\phi-B/\gamma}}.$$

Then you get time varying interest rates *and* FB regressions, full set of risk premiums! (But perfectly correlated with equity premium) We took it out. big mistake!

- Notice how we "distinguish intertemporal substitition from risk aversion"! $\gamma$ (low, 2) governs ies, whle $\gamma/S$ (high, $\bar{S} \approx 0.05$) governs precautionary savings (here) and risk aversion (in HJ bounds, etc). *you don't need state-nonseparable utility to distinguish ies from ra, this works just fine.*

- $$\begin{aligned} \frac{E_t(R^e_{t+1})}{\sigma_t(R^e_{t+1})} &= \frac{\sigma_t(M_{t+1})}{E_t(M_{t+1})} = \left(e^{\gamma^2\sigma^2[1+\lambda(s_t)]^2} - 1\right)^{\frac{1}{2}} \approx \gamma\sigma[1+\lambda(s_t)] \\ &= \frac{\gamma\sigma}{\bar{S}}\sqrt{1-2\left(s_t-\bar{s}\right)} \end{aligned}$$

  Again, $\gamma/\bar{S}$ controls this, like precautionary savings. Also you see how it rises as s declines, just as we hoped it would.

**Simulation and calibration**

$$\frac{P_t}{C_t}(s_t) = E_t\left[M_{t+1}\frac{C_{t+1}}{C_t}\left(1+\frac{P_{t+1}}{C_{t+1}}(s_{t+1})\right)\right]$$

- Parameters

- Steady state spc distribution. Note that it is left skewed. We plot other variables as functions of this state variable, its AR(1) and steady state then tell you how other things evolve.

- P/C Figure 3 Almost linear, not quite. Thus, pc AR is almost exactly that of S. PC reveals S in this model

- Figure 4, 5, 6. Conditional mean variance and sharpe. Note conditional variance *is* higher in recessions, just not as much as conditional means, so Sharpe ratios also rise. Conditional variance higher now!

- Simulated data Table 2. Note variance of return and p-d – driven all by 1.5 variance of consumption. risk aversion amplifies.

- Autocorrelations and cross correlations. Table 3, 4. Part of what the paper does is show that all these statistics reflect time varying risk aversion. Note the absolute vs level, this is a sign of garch.

- Table 5,6, long horizon regressions and variance decompositions

- Figure 7,8. Conditional capms, correlation etc. At any date, a one shock model. Yet unconditional correlations are smaller. And more when you time aggregate.

- Table 7. Really cool. Notice the are correlations about the same as in the data! Notice the apparent cross-correlation from returns to consumption growth!

- Table 8. The static CAPM is a better approximation than the true consumption CAPM. (emphasize that there is an exact conditional consumption capm driving the data). Why? In low S times, both returns and M are more sensitive to a consumption shock.

- History plots.

**Long run equity premium**

$$M_{t,t+k} = \delta^k \left( \frac{S_{t+k}}{S_t} \frac{C_{t+k}}{C_t} \right)^{-\gamma}.$$

In one period S moves one for one with C, and "amplifies".

As we go to longer horizons, S and C become uncorrelated. Thus "fear of recession" not "fear of consumption decline" become separate events, and "fear of recession" is stronger.

But $S$ is stationary. Why aren't we back to consumption in the long run?

Answer $S^{-\gamma}$ is not stationary! Fear of occasional deeper and deeper recessions builds with horizon.

**Macro and nonstochastic analysis**

$$r_t^f = -\ln(\delta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma^2 \sigma^2}{2} \left[1 + \lambda(s_t)\right]^2,$$

Don't leave off the second term! If you do, why isn't Rf varying a lot?

Also $E(\Delta c) = 1.89\%$, $r = 0.94\%$. Infinite prices!

**Internal vs. external habit is not a big deal**

Figure 10

$$MU_t = \frac{\partial U_t}{\partial C_t} = (C_t - X_t)^{-\gamma} - E_t \left[ \sum_{j=0}^{\infty} \delta^j \left(C_{t+j} - X_{t+j}\right)^{-\gamma} \frac{\partial X_{t+j}}{\partial C_t} \right]$$

Point: Asset pricing only depends on *ratios* of marginal utility. If the "internal" effect just raises all marginal utilities, it has no effect at all.

**Example:**

Suppose habit accumulation is linear, and there is a constant riskfree rate or linear technology equal to the discount rate, $R^f = 1/\delta$. The consumer's problem is then

$$\max \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} \quad s.t. \sum_t \delta^t C_t = \sum_t \delta^t e_t + W_0; \;\; X_t = \theta \sum_{j=1}^{\infty} \phi^j C_{t-j}$$

The first order conditions are

$$MU_t = E_t \left[ MU_{t+1} \right]$$

In the external case, marginal utility is simply

$$MU_t = (C_t - X_t)^{-\gamma}.$$

In the internal case, marginal utility is

$$MU_t = (C_t - X_t)^{-\gamma} - \theta \sum_{j=1}^{\infty} \delta^j \phi^j E_t \left(C_{t+j} - X_{t+j}\right)^{-\gamma}$$

The sum measures the habit-forming effect of consumption. Now, guess the same solution as for the external case,

$$(C_t - X_t)^{-\gamma} = E_t \left[(C_{t+1} - X_{t+1})^{-\gamma}\right].$$

and plug in.We find that the internal marginal utility is simply proportional to externa marginal utility

$$MU_t = \left(1 - \frac{\theta\delta\phi}{1 - \delta\phi}\right)(C_t - X_t)^{-\gamma}.$$

Since this expression satisfies the first order condition $MU_t = E_t MU_{t+1}$, we confirm the guess. Ratios of marginal utility are the same, so allocations and asset prices are completely unaffected by internal vs. external habit in this example.