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Appendix B from John H. Cochrane, “Determinacy and Identification with Taylor Rules”

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This appendix contains additional literature review and many extensions left out of the text for reasons of space.

Sections A and B critically review more of the literature on identification and determinacy in new-Keynesian and related models. If you want to know “what about *x*’s approach to these issues?” you are likely to find an answer here.

In particular, Section A.1 reviews the learnability criterion advanced by McCallum (2003, 2009). Section A.2 reviews recent proposals by Loisel (2007) and Adão, Correia, and Teles (2011), which seem to solve the multiple-equilibrium question. In fact, they propose a limiting case with infinite eigenvalues, a subset of the timing issues reviewed below. Section A.3 reviews Atkeson, Chari, and Kehoe (2010).

Section B on identification acknowledges the many papers that have made related points critical of identification in new-Keynesian models. It also tackles some attempts to overcome the identification problems. Lubik and Schorfheide (2004) try to identify the region—determinacy versus indeterminacy—without having to measure specific parameters using likelihood-based measures. I show generally that their identification comes from restrictions on the lag length of the unobservable shocks.

Section B.3 shows that in general one sets to zero movement in eigenvectors corresponding to exploding eigenvalues, so the latter cannot be measured. Thus, the problems are not specific to the structure of the three-equation new-Keynesian model.

Section B.4 reviews identification in full-system approaches, that is, complete models. Since none of those efforts have been focused on assessing determinacy, I do not cover them in the main text. For now, despite identification problems, the models impose eigenvalues greater than one and study dynamics.

Section C collects some extensions of the frictionless model. Section C.1 explores identification in all the equilibria of the frictionless model. I show that for $\phi < 1$, we still cannot identify ϕ separately from ρ. For $\phi > 1$ and with a prior that $\rho < 1$, we can identify ϕ for every equilibrium except the new-Keynesian choice. If the economy does explode, we can measure the speed of that explosion.

Section C.2 explores the impulse-response functions of the simple model. The new-Keynesian model produces responses by supposing that endogenous variables jump to a different equilibrium. The subsection contrasts the response function in the new-Keynesian equilibrium, which combines an (implicit) fiscal shock with a monetary policy shock, to the response function in the active-fiscal equilibrium of the same model, which has a monetary policy shock with no concurrent fiscal shock.

Section D addresses the question, what happens if you run Taylor rule regressions in artificial data from new-Keynesian models? As a particular example, if you run the regressions of the first half of Clarida, Galí, and Gertler’s (2000) paper on data from the model presented in the second half of the paper, do the regressions recover the Taylor rule parameters in the model? The answer is no. The text answered this question in the context of the simple model of Section II: we saw that if we ran the Taylor rule regression in data generated by the very new-Keynesian model, we would recover the shock autocorrelation process, not the Taylor rule parameter. This section answers the same question in the context of a three-equation model.

This section also answers the question, If not a change in the Taylor rule, what did Clarida et al. measure? The right answer is really that it’s a mongrel coefficient; it doesn’t matter. However, this section also gives an example of a shift in policy that could cause the measured Taylor coefficient to rise spuriously. In my example, the Taylor rule coefficient is constant at $\phi = 1.1$, but the Fed gets better at offsetting IS shocks, that is, following better the “natural rate.” This causes measured Taylor rule coefficients to rise as they do in the data. This example also answers a natural generalization of the frictionless model by adding an IS shock.

Section E addresses leads and lags in Taylor rules in the context of the simple frictionless model, continuous-time models, and the three-equation model. It turns out that determinacy questions depend quite sensitively on the timing assumptions in the Taylor rule. This contrasts with the usual feeling that Taylor rules are robust to such timing assumptions. The problem is particularly evident on taking the continuous-time limit. This section also addresses a natural question, what if we change the timing conventions from the models of the text?

Section D gives analytic solutions to the standard three-equation model. These models are often solved numerically, but then it is hard to know how specific parameters enter, especially for identification questions. The solutions are algebraically laborious, so I hope documenting them here is independently useful. In particular, I document analytic expressions for the eigenvalues, allowing analysis of the regions of determinacy and indeterminacy. I also fully solve the models with AR(1) shocks, which allows us to examine response functions and dynamics. In Section E.4, I verify the claim made in the text that zero-gap equilibrium, though achievable with a stochastic intercept interest rate policy that directly offsets shocks, is not achievable via a Taylor rule.

A. Related Literature on Determinacy and Equilibrium Selection

The question, what determines the price level, especially in an economy with fiat money and an interest rate target, naturally has a long history in economics, which I cannot begin to review. Patinkin (1949, 1965) brought price-level indeterminacy questions to the fore in postwar macroeconomics. Sargent and Wallace (1975) present the standard proof that interest rate targets lead to indeterminacy. McCallum (1981) was the first to suggest that an interest rate target that varied with economic conditions might overturn Sargent and Wallace's result. I will not even try to cite the literature on fiscal foundations of price-level stability. Cochrane (2005) contains one review. Sims (1994, 381) encapsulated the basic point well: "The existence and uniqueness of the equilibrium price level cannot be determined from knowledge of monetary policy alone; fiscal policy plays an equally important role." Sims also said, "these points are not new."

1. Learnability

In a series of papers, summarized in McCallum (2003), McCallum argues for a "minimal state value" (MSV) criterion to pick from multiple equilibria. In my examples, this criterion rules out the explosive solutions (which depend on initial inflation) or the sunspot solutions. However, this seems a philosophical rather than an economic criterion. Addressing this criticism, McCallum proposes instead that one choose equilibria by whether they are "learnable" or not, in the sense of Evans and Honkapohja (2001), and argues that one can derive the MSV criterion from this consideration. Woodford (2003) in commenting on McCallum disagrees and charges that the "wrong" equilibrium is often the learnable one.

McCallum (2003, 1154) explicitly states that his proposals do not apply to selecting among nominal indeterminacies and apply only to models with multiple real paths. Therefore, it appears, he would not apply them to the frictionless models on which I have focused. However, he does analyze the linearized three-equation model, presumably because in this case nominal indeterminacy spills over to real variables. Furthermore, his analysis (1160) is confined to the multiple solutions that emerge when the Taylor principle is not satisfied, that is, $\phi < 1$. He argues that none of these solutions are learnable. To him, then, the point of the Taylor rule is to make the forward-looking solution learnable: "the Taylor principle is of importance because its non-satisfaction leads to a situation in which all [rational expectations] solutions fail to be learnable." He does not address the multiple explosive solutions that occur when the Taylor principle does apply ($\phi > 1$).

McCallum (2009) is a new and particularly clear example of this line. McCallum argues in the context of the simple model of Section II that the explosive equilibria are not "learnable" and the unique bounded equilibrium is the only "learnable" one. This argument applies to nominal indeterminacies. However, I think he got it backward (Cochrane 2009). McCallum assumed that the public can directly observe the monetary policy shock x_t. When we make the opposite assumption, that the policy disturbance is not directly observable, so that agents must run regressions to measure it, I obtain the opposite result: the *explosive* equilibria are learnable and the unique local equilibrium is the only one that is *not* learnable.

This result is closely tied to identification. If an econometrician cannot identify ϕ, how is the public supposed to learn it? In these models the public and econometricians have the same information sets. To measure the shock you need to know the slope coefficient, so ϕ and x_t are the same question. However, as shown in Section III.A, ϕ is identified in the $\phi > 1$, explosive equilibria. When the economy does explode, you can measure how fast it does so. Econometricians can learn ϕ very quickly in this case, in line with Woodford's (2003) general comment: the "wrong" equilibria are the "learnable" ones.

More generally, even if true, I think this is a last gasp. Is inflation really determined at a given value because for any other value the Fed threatens to take us to a valid but "unlearnable" equilibrium? Why should we care about such a threat?

2. *Interest Rate Rules That Seem to Work* (*Infinite Eigenvalues*)

Loisel (2007) proposes a rule that responds to both current and future inflation (simplified to this setting),

$$i_t = r + E_t\pi_{t+1} + \psi(\pi_t - z_t),$$

where ψ is any nonzero constant, and z_t is any exogenous random variable. If we merge this rule with the usual Fisher equation

$$i_t = r + E_t\pi_{t+1}, \tag{B1}$$

we obtain a unique equilibrium:

$$\pi_t = z_t.$$

This seems to be the Holy Grail: a nominal interest rate rule that delivers a unique equilibrium inflation rate in a frictionless economy. The trick, as Loisel explains, is to have the interest rate rule exactly cancel the troublesome forward-looking terms of the model.

To digest this proposal, write it as a special case of a rule that responds to current and expected future inflation,

$$i_t = r + \phi_0\pi_t + \phi_1 E_t\pi_{t+1} - \phi_0 z_t.$$

Merging this rule with the usual Fisher equation (B1), we obtain

$$E_t\pi_{t+1} = \phi_0\pi_t + \phi_1 E_t\pi_{t+1} - \phi_0 z_t,$$

$$E_t\pi_{t+1} = \frac{\phi_0}{1-\phi_1}\pi_t - \frac{\phi_0}{1-\phi_1}z_t.$$

As usual, this system displays multiple equilibria:

$$\pi_{t+1} = \frac{\phi_0}{1-\phi_1}\pi_t - \frac{\phi_0}{1-\phi_1}z_t + \delta_{t+1}; \quad E_t\delta_{t+1} = 0. \tag{B2}$$

The eigenvalue or root is $\phi_0/(1-\phi_1)$. Thus, if

$$\frac{\phi_0}{1-\phi_1} > 1, \tag{B3}$$

we have at least a unique locally bounded equilibrium,

$$\pi_t = E_t\sum_{j=0}^{\infty}\left(\frac{1-\phi_1}{\phi_0}\right)^j z_{t+j}.$$

Condition (B3) is equivalent to

$$\phi_0 + \phi_1 > 1,$$

which has the familiar Taylor rule ring to it, that overall interest rates must rise more than one-for-one with inflation.

Now, we can study not only the *point* $\phi_1 = 1$ but the *limit* $\phi_1 \to 1$. As $\phi_1 \to 1$, the eigenvalue or root (B3) of the difference equation (B2) rises to infinity. For ϕ_1 near one, the Fed is saying, "if inflation doesn't come out to the desired value, we'll hyperinflate very fast." In the limit, multiple equilibria are ruled out by a threat to hyperinflate with infinite speed. Thus, this proposal is an accelerated version of the usual logic. That is not a criticism: much analysis of Taylor rules finds that larger responses are better, and you cannot get larger than infinite. The point is just that we can digest this proposal as a limit of the usual logic rather than have to think of it as a fundamentally new type of interest rate target.

We can also see that it is a knife-edge case of the fact, studied below, that determinacy in new-Keynesian models depends sensitively on the timing assumptions. A coefficient even ε different from $\phi_1 = 1.000$ or a time index slightly different from $+1.000$ brings us back to the usual world.

Finally, all forward-looking rules, and this one in particular, require the central bank to respond to private expectations. In a rational expectations equilibrium, expectations $E_t(P_{t+1})$ or $E_t(\pi_{t+1})$ are just given by $E_t(z_{t+1})$. However, if people develop a sunspot expectation for more inflation, the Fed must increase interest rates to match it. One can easily question the informational and off-equilibrium or game-theoretic foundations of such a response.

Adão et al. (2011) advance a similar proposal. Simplified to the linearized, constant real-rate, frictionless environment, they propose the target

$$i_t = r + E_t p_{t+1} - z_t, \tag{B4}$$

where z_t is any exogenous random variable. If we merge this rule with the usual Fisher equation expressed as

$$i_t = r + E_t(p_{t+1} - p_t), \tag{B5}$$

we obtain a unique equilibrium

$$p_t = z_t.$$

Thus we have an interest rate target that delivers a unique, determinate, price level in a frictionless economy, with no multiple equilibria. Again, the key is that the interest rate rule exactly cancels the forward-looking term of the model, in this case the price level rather than the inflation rate. Adão et al.'s analysis is in fact conducted in the full nonlinear version of a cash-in-advance model with labor supply, so linearization, local approximation, and a frictionless economy are not central to the result.[2]

We can understand this proposal as a similar infinitely explosive limit of the sort of "Wicksellian" price-level-stabilizing interest rate rules studied by Woodford (2003, 81). Generalize the rule to

$$i_t = r + \phi_1 E_t p_{t+1} + \phi_0 p_t - z_t.$$

Equate to the Fisher equation (B5), and we find the equilibrium condition

$$E_t p_{t+1} = \frac{1 + \phi_0}{1 - \phi_1} p_t + \frac{1}{1 - \phi_1} z_t.$$

The eigenvalue is

$$\lambda = \frac{1 + \phi_0}{1 - \phi_1},$$

and we have a unique locally bounded equilibrium if $\|\lambda\| > 1$. Woodford studies the case $\phi_1 = 0$ and so obtains the condition $\phi_0 > 0$. Adão et al. specify $\phi_0 = 0$ and study the limit as $\phi_1 \to 1$. Again, you can see that this is the limit of an infinite eigenvalue, in which the threatened explosion happens infinitely fast.

Loisel's (2007, 11) actual example of an interest rate rule that seems to avoid multiple equilibria is given in the context of the three-equation model; in my notation,

$$i_t = E_t \pi_{t+1} + \phi_{\pi,0} \pi_t + \frac{1}{\sigma}(E_t y_{t+1} - y_t).$$

[2] Here's the nonlinear version: Start with the consumer's first-order condition,

$$\frac{u_C(t)}{P_t} = (1 + i_t) E_t\left[\frac{\beta u_C(t+1)}{P_{t+1}}\right].$$

Assume a constant endowment $C_t = Y$, so the u_C terms cancel. Then, we can write

$$1 + i_t = \frac{1}{P_t E_t[\beta / P_{t+1}]}.$$

Write the policy rule

$$1 + i_t = \frac{1}{z_t E_t[\beta / P_{t+1}]}.$$

The globally unique equilibrium is $P = z_t$.

If we place this rule in the standard model (26)–(28),

$$y_t = E_t y_{t+1} - \sigma(i_t - E_t \pi_{t+1}),$$

$$\pi_t = \beta E_t \pi_{t+1} + \gamma y_t,$$

we can quickly see that $\pi_t = 0$, $y_t = 0$ is the only equilibrium as long as $\phi_{\pi,0} \neq 0$. In this context, the variables are deviations from a desired equilibrium, so we have shown that the interest rate rule implements the desired equilibrium uniquely.

We can understand this rule as the limit of a standard rule of the form

$$i_t = \phi_{\pi,0}\pi_t + \phi_{\pi,1}E_t\pi_{t+1} + \phi_{y,0}y_t + \phi_{y,1}y_{t+1}.$$

When $\phi_{y,1} = 1/\sigma$, the (single, repeated) eigenvalue of the three-equation model, derived below, is

$$\lambda = \frac{1 + \sigma(\phi_{y,0} + \gamma\phi_{\pi,0})}{\beta + \sigma[\gamma(1 - \phi_{\pi,1}) + \beta\phi_{y,0}]}.$$

Taking the limit, $\phi_{\pi,1} \to 1$, $\phi_{y,1} = 1/\sigma$, $\phi_{y,0} = -1/\sigma$, the denominator goes to zero, so again this is a limit with an infinite eigenvalue.

3. Atkeson, Chari, and Kehoe: Blowing Up the World for a Day

Atkeson et al. (2010, 50) eloquently criticize “implementation via nonexistence” or blow-up-the-world threats. However, their “sophisticated policies” also rely on policy settings for which first-order conditions do not hold, so no equilibrium is possible. They blow up the world for only one period; after that period a competitive equilibrium remains possible.

Here is a simple version of the Atkeson et al. model that explains these points. (Relative to their model starting on p. 53, I take $\gamma = 0$ and $y =$ constant, or $\psi = 0$.) Specify an endowment economy with constant consumption so that the linearized Euler/IS equation is

$$i_t = E_t\pi_{t+1}. \tag{B6}$$

Add an interest-inelastic money demand function with a money demand shock v_t so that money growth μ_t and inflation π_t follow

$$\mu_t = \pi_t + v_t. \tag{B7}$$

The central bank can set either i_t or μ_t at the beginning of period t as functions of time $t-1$ information.

Suppose that the central bank follows an interest rate rule

$$i_t = \phi E_{t-1}(\pi_t).$$

(The model has a set of producers who announce prices x_t one period in advance, according to $x_t = E_{t-1}\pi_t$, so the Fed can see $E_{t-1}\pi_t$ directly.) Money growth will then be endogenous, satisfying (B7). This is the same model as Section II with a money demand function and an inconsequential change in timing convention. Its equilibria are fully described by one condition:

$$E_t\pi_{t+1} = \phi E_{t-1}(\pi_t). \tag{B8}$$

The equilibrium $\pi_t = 0$ is possible. But many other equilibria are possible too, indexed by alternative initial values of inflation and by “sunspot” shocks. Any process

$$\pi_t = \phi\pi_{t-1} + \delta_t; \quad E_{t-1}(\delta_t) = 0$$

is an equilibrium. The Taylor rule $\phi > 1$ means that any equilibrium other than $\pi_t = 0$ eventually leads to hyperinflation or deflation, but nothing in the model as it stands rules that out (“pure interest rate rules,” p. 63).

By contrast, if the central bank follows a money-targeting rule $\mu_t = 0$, inflation is uniquely determined at $\pi_t = -v_t$. Interest rates are then market determined at $i_t = E_t\pi_{t+1} = -E_t v_{t+1}$. The money demand shock v_t serves only to motivate why an interest rate rule might be desirable to avoid inflation volatility, so I will drop it and specify $v_t = 0$ in what follows.

Here is Atkeson et al.’s main idea (“reversion to a hybrid rule,” p. 65). Like them, I simplify to a perfect-foresight version of the model, though I retain the E_t notation for clarity. The Fed follows the interest rate rule $i_t = \phi E_{t-1}\pi_t$ with $\phi > 1$, as long as expected inflation is below some value $\bar{\pi}$ or above some value $\underline{\pi}$. If expected inflation $E_{t-1}\pi_t$ gets out of this range, the Fed switches to the money growth rule $\mu_t = 0$. The economy then reverts to $\pi_t = 0$, and an equilibrium forms on each date after that point.

Inflations or deflations are therefore stabilized, but how does this provision rule out inflationary paths? Let T denote the date of the switch, so $\mu_t = \pi_t = 0$ for all $t \geq T$, but π_{T-1} is still large. Everyone at time $T-1$ and before knew this would happen, so the first-order condition requires $i_{T-1} = E_{T-1}(\pi_T) = 0$. However, this requirement conflicts with the policy rule $i_{T-1} = \phi E_{T-2}(\pi_{T-1})$, which is still a large number. Hence, the path cannot be an equilibrium. (This is their proof, bottom of p. 66 and top of p. 67.)

For one period, the Fed follows a Taylor rule demanding a high interest rate $i_{T-1} = \phi E_{T-2}(\pi_{T-1})$ and a money growth rule $\mu_T = 0$, which demands a low interest rate $i_{T-1} = E_{T-1}(\pi_T) = 0$, at the same time. It is a blow-up-the-world policy, just for a single period. The Fed cannot do that in markets, so it is an impossible policy commitment by usual Ramsey rules. And it is a choice. The Fed can stop the inflation, allowing an equilibrium to form at each date, by just waiting one period to switch to a money growth rule, so it is not trying simultaneously to run a money growth rule and a Taylor rule.

Chari et al.'s prose is not always clear on this point, but I can find agreement here: "our definition does not require that, when there is a deviation in period t, the entire sequence starting from period 0, including the deviation in period t, constitute a period-zero competitive equilibrium. Indeed, if we achieve unique implementation, then such a sequence will not constitute a period-zero equilibrium" (60). The only argument really is whether trying to run a policy that requires two different values of the interest rate for one period is as "dire" (50) as a hyperinflation might be.

Minford and Srinivasan (2010) similarly rule out equilibria by having the central bank switch to a money growth rule for large values of inflation or deflation—but at the same time maintain the Taylor rule for interest rates.

B. Related Literature on Identification

The point that out-of-equilibrium or alternative-equilibrium behavior cannot be measured from data in a given equilibrium is well known, seemingly obvious, once stated, and applies broadly in macroeconomics. Among many others, Sims (1994, 384) states as one of four broad principles, "Determinacy of the price level under any policy depends on the public's beliefs about what the policy authority would do under conditions that are never observed in equilibrium." Cochrane (1998) shows analogously that one cannot test the off-equilibrium government behavior that underlies the fiscal theory of the price level: Ricardian and non-Ricardian regimes make observationally equivalent predictions for equilibrium time series without further assumptions. (The models may make very different response function and policy predictions, however.) The point that crucial estimates of many macroeconomic models hinge on "incredible" identification assumptions goes back at least to Sims (1980). My contribution is to apply these well-known principles to new-Keynesian models.

1. Literature on Lack of Identification

The papers closest to this one are Beyer and Farmer (2004, 2007). Beyer and Farmer compare an "indeterminate" AR(1) model

$$p_t = aE_t(p_{t+1})$$

with $\|a\| < 1$ to a "determinate" AR(2),

$$p_t = aE_t(p_{t+1}) + bp_{t-1} + v_t,$$

where they choose a and b so that one root is stable and the other unstable. Both models have AR(1) representations, so there is no way to tell them apart. They conjecture on the basis of this result that Lubik and Schorfheide (2004) attain identification by lag length restrictions.

Beyer and Farmer (2004) compute solutions to the three-equation new-Keynesian model. They note that the equilibrium dynamics are the same for any value of the Fed's Taylor rule coefficient on inflation, as long as that coefficient is greater than one. Thus, they see that the Taylor rule coefficient is not identified by the equilibrium dynamics. They examine the model

$$u_t = E_t u_{t+1} + 0.005(i_t - E_t\pi_{t+1}) - 0.0015 + v_{1t},$$

$$\pi_t = 0.97E_t\pi_{t+1} - 0.5u_t + 0.0256 + v_{2t},$$

$$i_t = 1.1E_t\pi_{t+1} + 0.028 + v_{3t},$$

where v_{it} are i.i.d. shocks. They compute the equilibrium dynamics ("reduced form") as

$$\begin{bmatrix} u_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.02 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0.05 \\ -0.5 & 1 & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{bmatrix}. \tag{B9}$$

They state that "all policies of the form

$$i_t = -f_{32}E_t[\pi_{t+1}] + c_3 + v_{3t},$$

for which

$$|f_{32}| > 1$$

lead to exactly the same reduced form ... as long as c_3 and f_{32} are chosen to preserve the same steady state interest rate" (24). They do not state whether this is an analytical result or simply the result of trying a lot of values; since the computation of (B9) is numerical, one suspects the latter.

Davig and Leeper (2007) calculate an economy in which the Taylor rule stochastically shifts between "active" $\phi > 1$ and "passive" $\phi < 1$ states. They show that the system can display a unique locally bounded solution even though one of the regimes is passive. Intuitively, we can rule out a value of inflation if it will lead to a future explosion after a stochastic shift to a new regime, even if it does not lead to an explosion as long as the current regime is in place. Even if one could identify and measure the parameters of the Taylor rule, this model argues against the stylized history that the United States moved from "passive" and hence "indeterminate" monetary policy in the 1970s to an "active" and hence "determinate" policy in the 1980s. As long as agents understood some chance of moving to an active policy, inflation was already determinate in the 1970s.

Woodford (2003) notices the identification problem. He discusses Taylor's (1999) and Clarida et al.'s (2000) regression evidence that the Fed responded less than one-to-one to inflation before 1980 and more than one-to-one afterward. He writes, "Of course, such an interpretation depends on an assumption that the interest rate regressions of these authors correctly identify the character of systematic monetary policy during the period. In fact, an estimated reaction function of this kind could easily be misspecified" (93). An example in which the measured ϕ coefficient is one-half of the true value follows. However, though Woodford sees the possibility of a bias in the estimated coefficients, he does not say that the structural parameter ϕ is unidentified.

Minford, Perugini, and Srinivasan (2002, forthcoming) address a related but different identification point: Does a Taylor rule regression of interest rates on output and inflation establish that the Fed is in fact following a Taylor rule? The answer is no: Even if the Fed targets the money stock, equilibrium nominal interest rates, output, and inflation will vary, so we will see a "Taylor rule" type relation. As output rises or inflation rises with a fixed money stock, money demand rises, so equilibrium interest rates must rise. As a very simple explicit example, consider a constant money supply equal to money demand,

$$m_t^d - p_t = \alpha y_t - \beta i_t,$$

$$m_t^d = m^s.$$

In equilibrium, we see a Taylor-like relation between nominal interest rates, output, and the price level:

$$i_t = -\frac{1}{\beta}m^s + \frac{\alpha}{\beta}y_t + \frac{1}{\beta}p_t.$$

This is an important point: just because the central bank says that it is following an inflation target and just because its short-run operating instrument is obviously an interest rate does not by itself document that the central bank is not paying attention to a monetary aggregate or that price-level determinacy does not in the end really come from such a target.

Mavroeidis (2004, 2005) argues that Clarida el.'s identification is "weak" because when inflation is well controlled, there is little variation in the right-hand variable. I go further to argue that identification is absent, because there is no variation in the crucial right-hand variable, which is the deviation of inflation from equilibrium.

2. Lubik and Schorfheide: Testing Regions

Lubik and Schorfheide (2004) try to identify the region—determinacy versus indeterminacy—without having to measure specific parameters. Alas, their identification comes from restrictions on the lag length of the unobservable shocks. Beyer and Farmer (2007) make this point with a series of examples. The following discussion presents the general case.

Identification is different from approximation. We can be forgiven for running finite-order VARs when theory does not restrict lag lengths. As data increase, lag lengths can increase, and we slowly approach correct estimates. When we use a false restriction to generate identification, there is no sense in which the answer is approximately right or ever gets better as we increase sample size. (Sims [1980, 5, n. 5] makes this point eloquently.)

Lubik and Schorfheide explain their ideas in the same single-equation setup as in Section II, simplifying even further by assuming a white-noise monetary policy disturbance, $\rho = 0$. The equilibrium is characterized again by (4), which becomes

$$E_t \pi_{t+1} = \phi \pi_t + \varepsilon_t. \tag{B10}$$

The solutions are, generically,

$$\pi_{t+1} = \phi \pi_t + \varepsilon_t + \delta_{t+1},$$

where δ_{t+1} represents the inflation forecast error. If $\phi > 1$, the unique locally bounded solution is

$$\pi_t = -\frac{\varepsilon_t}{\phi}.$$

If $\phi < 1$, then any δ_{t+1} with $E_t \delta_{t+1} = 0$ gives rise to a locally bounded equilibrium.

Lubik and Schorfheide agree that ϕ is not identified when $\phi > 1$. For example, the likelihoods in their figure 1 are flat functions of ϕ for the region $\phi > 1$. However, they still claim to be able to test for determinacy—to distinguish the $\phi > 1$ and $\phi < 1$ regions. The essence of their test is a claim that the model with indeterminacy $\phi < 1$ can produce time-series patterns that the model with determinacy cannot produce.

They explain the result with this simple example. Since δ_{t+1} is arbitrary, it does no harm to restrict $\delta_{t+1} = M\varepsilon_{t+1}$, with M an arbitrary parameter. In this example, then, the (local or bounded) solutions are

$$\begin{aligned} \phi > 1: &\quad \pi_t = -\frac{\varepsilon_t}{\phi}, \\ \phi < 1: &\quad \pi_t = \phi \pi_{t-1} + \varepsilon_{t-1} + M\varepsilon_t. \end{aligned} \tag{B11}$$

If $\phi > 1$, the model can produce only white-noise inflation π_t. If $\phi < 1$, the model produces an ARMA(1, 1) in which ϕ is identified as the autoregressive root. Thus, if you saw an ARMA(1, 1), you would know you are in the region of indeterminacy. They go on to construct a likelihood ratio test for determinacy versus indeterminacy.

Alas, this identification is achieved only by restricting the nature of the shock process x_t. If the shock process x_t is not white noise, then the $\phi > 1$ solution can display complex dynamics in general and an ARMA(1, 1) in particular. Since the shock process is unobserved, we cannot in fact tell even the region $\phi > 1$ from the region $\phi < 1$. I can sum up this point in a proposition.

Proposition. For any stationary time-series process for $\{i_t, \pi_t\}$ that represents an equilibrium of (B10) and for any $\tilde{\phi}$, one can construct an x_t process that generates the same process for the observables $\{i_t, \pi_t\}$ as an equilibrium of (B10) using the alternative $\tilde{\phi}$. If $\tilde{\phi} > 1$, the observables are generated as the unique bounded forward-looking solution. Given an assumed $\tilde{\phi}$ and the process $\pi_t = a(L)\varepsilon_t$, we construct $x_t = b(L)\varepsilon_t$ with

$$b_j = a_{j+1} - \tilde{\phi} a_j \tag{B12}$$

or, in lag operator notation,

$$b(L) = (L^{-1} - \tilde{\phi})a(L) - a(0)L^{-1}. \tag{B13}$$

In particular, any observed time-series process for $\{i_t, \pi_t\}$ that is consistent with a $\phi < 1$ model is also consistent with a different $\tilde{\phi} > 1$ model. In the absence of restrictions on the unobserved forcing process $\{x_t\}$, there is no way to tell the regime with determinacy from the regime with indeterminacy. Equivalently, the joint set of parameters including ϕ and the parameters of the x_t process are unidentified; one can identify only some of these parameters, for example, $\phi < 1$ versus $\phi > 1$, by fixing others, for example, the parameters of x_t.

Proof. Start with any process for inflation $\pi_t = a(L)\varepsilon_t$. Choose an arbitrary $\tilde{\phi} > 1$. Then, we construct a disturbance process $x_t = b(L)\varepsilon_t$ so that the forward-looking equilibrium with arbitrary $\tilde{\phi} > 1$ generates the desired time-series process for inflation, that is,

$$\pi_t = a(L)\varepsilon_t = -E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} x_{t+j} = -E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} b(L)\varepsilon_{t+j}.$$

It is easy enough to check that (B12) is correct:

$$\begin{aligned}
-E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} b(L)\varepsilon_{t+j} &= -E_t \sum_{j=0}^{\infty} \frac{1}{\tilde{\phi}^{j+1}} \sum_{k=0}^{\infty} (a_{k+1} - \tilde{\phi}a_k)\varepsilon_{t+j-k} \\
&= -\frac{1}{\tilde{\phi}}[(a_1 - \tilde{\phi}a_0)\varepsilon_t + (a_2 - \tilde{\phi}a_1)\varepsilon_{t-1} + (a_3 - \tilde{\phi}a_2)\varepsilon_{t-2} + \cdots] \\
&\quad - \frac{1}{\tilde{\phi}^2}[(a_2 - \tilde{\phi}a_1)\varepsilon_t + (a_3 - \tilde{\phi}a_2)\varepsilon_{t-1} + (a_4 - \tilde{\phi}a_5)\varepsilon_{t-2} + \cdots] \\
&\quad - \frac{1}{\tilde{\phi}^3}[(a_3 - \tilde{\phi}a_2)\varepsilon_t + (a_4 - \tilde{\phi}a_3)\varepsilon_{t-1} + (a_5 - \tilde{\phi}a_4)\varepsilon_{t-2} + \cdots] + \cdots \\
&= a_0\varepsilon_t + a_1\varepsilon_{t-1} + a_2\varepsilon_{t-2} + \cdots.
\end{aligned}$$

If we choose a $\tilde{\phi} < 1$, then the construction is even easier. The solutions to (4) are

$$\pi_{t+1} = \tilde{\phi}\pi_t + x_t + \delta_{t+1},$$

where δ_t is an arbitrary unforecastable shock. To construct an x_t, we need therefore

$$(1 - \tilde{\phi}L)\pi_{t+1} = x_t + \delta_{t+1},$$

$$(1 - \tilde{\phi}L)a(L)\varepsilon_{t+1} = b(L)\varepsilon_t + \delta_{t+1}.$$

Obviously, forecast errors must be equated, so we must have $\delta_{t+1} = a_0\varepsilon_{t+1}$. Then

$$(1 - \phi L)a(L)\varepsilon_{t+1} = b(L)\varepsilon_t + a_0\varepsilon_{t+1},$$

$$(1 - \phi L)a(L) = a_0 + Lb(L),$$

and (B13) follows. The term i_t is just given by $i_t = r + E_t(\pi_{t+1})$ and so adds nothing once we match π dynamics. QED

Example. Suppose that we generate data from the Lubik-Schorfheide example with $\phi < 1$; that is, $x_t = \varepsilon_t$ is i.i.d., and therefore π_t follows the ARMA(1, 1) process (B11),

$$\pi_t = \phi\pi_{t-1} + M\varepsilon_t + \varepsilon_{t-1} = (1 - \phi L)^{-1}(M + L)\varepsilon_t.$$

We can generate exactly the same solution from a model with arbitrary $\tilde{\phi} > 1$ if we let the policy disturbance x_t be an ARMA(1, 1) rather than restrict it to be white noise. Using (B13), we choose $x_t = b(L)\varepsilon_t$ with

$$b(L) = (L^{-1} - \tilde{\phi})(1 - \phi L)^{-1}(M + L) - L^{-1}M$$

or, multiplying by $1 - \phi L^{-1}$and simplifying,

$$(1 - \phi L)x_t = [(L^{-1} - \tilde{\phi})(M + L) - (1 - \phi L)L^{-1}M]\varepsilon_t,$$

$$(1 - \phi L)x_t = \{[1 + (\phi - \tilde{\phi})M] - \tilde{\phi}L\}\varepsilon_t,$$

$$x_t - \phi x_{t-1} = [1 + (\phi - \tilde{\phi})M]\varepsilon_t - \tilde{\phi}\varepsilon_{t-1};$$

that is, x_t follows an ARMA(1, 1).

3. General Case: System Nonidentification

One might suspect that these results depend on the details of the three-equation model. What if one specifies a slightly different policy rule or slightly different IS or Phillips curves? The bottom line is that when you estimate dynamics from stationary variables, you must find stable dynamics. You cannot measure eigenvalues greater than one. In the forward-looking bounded solution, shocks corresponding to eigenvalues greater than one are set to zero.

To study identification, I trace the standard general solution method, as in Blanchard and Kahn (1980), King and Watson (1998), and Klein (2000). The general form of the model can be written

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{C}\boldsymbol{\epsilon}_{t+1}, \tag{B14}$$

where $\mathbf{y}_t$ is a vector of variables, for example, $\mathbf{y}_t = [y_t\ \pi_t\ i_t\ x_{\pi t}\ x_{dt}]'$. By an eigenvalue decomposition[3] of the matrix $\mathbf{A}$, write

$$\mathbf{y}_{t+1} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}\mathbf{y}_t + \mathbf{C}\boldsymbol{\epsilon}_{t+1},$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues λ_i and $\mathbf{Q}$ is the corresponding matrix of eigenvectors.

Premultiplying (B14) by $\mathbf{Q}^{-1}$, we can write the model in terms of orthogonalized variables as

$$\mathbf{z}_{t+1} = \boldsymbol{\Lambda}\mathbf{z}_t + \boldsymbol{\xi}_{t+1},$$

where

$$\mathbf{z}_t = \mathbf{Q}^{-1}\mathbf{y}_t; \quad \boldsymbol{\xi}_{t+1} = \mathbf{Q}^{-1}\mathbf{C}\boldsymbol{\epsilon}_{t+1}.$$

Since $\boldsymbol{\Lambda}$ is diagonal, we can solve for each $\mathbf{z}_t$ variable separately. We solve the unstable roots forward and the stable roots backward:

$$\|\lambda_i\| > 1: \quad z_{it} = \sum_{j=1}^{\infty} \frac{1}{\lambda_i^j} E_t \xi^i_{t+j} = 0, \tag{B15}$$

$$\begin{aligned} \|\lambda_i\| < 1: \quad z_{it} &= \sum_{j=0}^{\infty} \lambda_i^j \xi^i_{t-j}, \\ z_{it} &= \lambda_i z_{it-1} + \xi_{it}. \end{aligned} \tag{B16}$$

Thus, we choose the unique locally bounded equilibrium by setting the explosive z_{it} variables and their shocks to zero.

Denote by $\mathbf{z}^*$ the vector of the $\mathbf{z}$ variables corresponding to eigenvalues whose absolute value is less than one in (B16), denote by $\boldsymbol{\xi}^*_t$ the corresponding shocks, denote by $\boldsymbol{\Lambda}^*$ the diagonal matrix of eigenvalues less than one in absolute value, and denote by $\mathbf{Q}^*$ the matrix consisting of columns of $\mathbf{Q}$ corresponding to those eigenvalues. Since the other z variables are all zero, we can just drop them and characterize the dynamics of the $\mathbf{y}_t$ by

$$\mathbf{z}^*_t = \boldsymbol{\Lambda}^*\mathbf{z}^*_{t-1} + \boldsymbol{\xi}^*_t,$$

$$\mathbf{y}_t = \mathbf{Q}^*_t\mathbf{z}^*_t.$$

[3] King and Watson (1998) and Klein (2000) treat more general cases in which $\mathbf{A}$ does not have an eigenvalue decomposition. This generalization usually is just a matter of convenience, e.g., whether one substitutes in variable definitions or leaves them as extra relations among state variables.

The roots $\|\lambda\|$ that are greater than one do not appear anywhere in these dynamics. Thus we obtain general statements of the identification lessons that applied to ϕ in the simple example: (1) *We cannot measure eigenvalues greater than one from the equilibrium dynamics of this model*. Equation (B15) shows why: (2) *There is no variation in the linear combinations of variables you need to measure* $\|\lambda\| > 1$. For this reason, (3) *the equilibrium dynamics are the same for every value of the eigenvalues supposed to be greater than one.* The latter statement includes values of those eigenvalues that are less than one. The equilibrium with λ greater than one and no shocks by the new-Keynesian equilibrium selection criterion is observationally equivalent to the same no-shock equilibrium with λ less than one.

This solution gives rise to more variables **y** than there are shocks, so it is stochastically singular. We have

$$\mathbf{z}_t = \mathbf{Q}^{-1}\mathbf{y}_t.$$

Since some $\mathbf{z}_t$ are zero, this relationship describes linear combinations of **y** that are always zero. However, not all elements of **y** are directly observable. The "stochastic singularity" then links endogenous observables (y, π, i) to disturbances (x_π, x_d). Similarly, the expectational errors in $\boldsymbol{\xi}_{t+1} = \mathbf{Q}^{-1}\mathbf{C}\boldsymbol{\epsilon}_{t+1}$ jump to offset any real shocks so that $\xi^i_{t+1} = 0$ for $\|\lambda_i\| \geq 1$ at all dates.

New-Keynesian models are engineered to have "just enough" forward-looking roots. In new-Keynesian models, some of the shocks are arbitrary forecast errors because some of the structural equations involve expectations; the model stops at $E_t y_{it+1} =$ something else. In this case the backward solution leads to indeterminacy since forecast errors can be anything. Hence, new-Keynesian models specify that some of the roots are explosive (forward looking) so that the forecast errors are uniquely determined and there is a unique local solution.

The only possibility to rescue identification in this context is if there are cross-equation restrictions: if we can learn the λ from the parts of the dynamics we can see. Nobody has traced this idea to see if the parameters of the Taylor rule, or more generally the parameters that control determinacy in more complex models, can be identified by cross-equation restrictions.

4. *System Identification*

Identifying parameters by estimating the whole system is a promising possibility, especially if one feels uncomfortable at the strong assumptions that need to be made for single-equation methods. We write down a complete model, we find dynamics of the observable variables, and we figure out if there are or are not multiple structural parameters corresponding to each possible set of equilibrium dynamics. Model fit can be measured by distance between model and data impulse-response functions or by the likelihood function. For determinacy questions, one can then address whether the whole model produces eigenvalues in the zone of determinacy, which is not just a function of Taylor rule parameters in complex models. Of course, full systems include specifications of the stochastic process of shocks, so one must be careful that identification does not come crucially from lag length restrictions.

There is now a quickly evolving literature on estimating fully specified new-Keynesian models and parallel investigation of identification in those estimates. Examples include Rotemberg and Woodford (1997, 1999, and esp. 1998 [with a focus on identification]), Smets and Wouters (2003), Christiano, Eichenbaum, and Evans (2005), Giannoni and Woodford (2005), and Ireland (2007).

Ontaski and Williams (2010), Féve, Matheron, and Poilly (2007), Canova and Sala (2009), and Iskrev (2010) question the identification in these estimates. The overall conclusion is that many parameters of typical large-scale models are poorly identified—likelihood functions and other objectives are flat—so some parameters must be fixed ex ante or by Bayesian priors so strong that large parameter regions are excluded. Likelihood functions and other objectives often have local minima raising the global identification issue.

For example, the difference between prior and posterior is a measure of how much the data have to say about a parameter. Tellingly, in Smets and Wouter's (2003) estimates, the prior and posterior for the inflation response of monetary policy ϕ_π are nearly identical (1147, fig. 1C), and the estimate is 1.68 relative to a prior mean of 1.70, suggesting that the policy rule parameters are at best weakly identified, even in a local sense.

Ontaski and Williams (2010) find that changing priors affects Smets and Wouter's structural parameter estimates substantially. They also find numerous local minima. They report that "although our parameter estimates differ greatly, the implied time series of the output gap that we find nearly matches that in SW and the qualitative features of many of the impulse responses are similar" (146). Canova and Sala (2009) use a similar numerical evaluation of identification in large-scale new-Keynesian models with similar results.

Ireland (2007) shows analytically that several parameters in a large-scale model are not identified. He also shows that it is basically impossible to distinguish econometrically between two versions of the model that provide very different interpretations of postwar U.S. monetary history.

None of these papers even asks whether there are equivalent parameters from the region of nondeterminacy that account equally for the observed dynamics. They simply rule out parameters from the nondeterminacy region ex ante or by strong Bayesian priors. The popular DYNARE computer programs will not allow one to compute a solution from this region. In sum, nobody has tried to exploit full-system identification to surmount the difficulties posed above.

This is not a criticism. The authors of these papers are not interested in testing for determinacy. None of them address Clarida et al.'s (2000) question, whether the Federal Reserve moved from an "indeterminate" regime before 1980 to a "determinate" one after that. More broadly, they are not interested in testing the new-Keynesian model, asking whether some other model might equally account for the data. They are interested in matching dynamics of output, inflation, and other variables, by elaboration of the basic model, imposing determinacy where there is any question, and making arbitrary choices of parameters when those are weakly identified. Lack of identification, as expressed by Ontaski and Williams (2010), is almost a feature, not a bug, as it means that the model's ability to match dynamics is "robust" to parameter choices, though all recognize that policy analysis depends on poorly identified parameters.

Perhaps in the future the testing issue will resurface, and then we can evaluate whether the identifying assumptions are reasonable.

5. *Identification via Impulse-Response Functions*

We can get a more concrete sense of these issues by looking at the impulse-response functions of a fully specified model. Below, I find the full solution of the model

$$
\begin{aligned}
y_t &= E_t y_{t+1} - \sigma r_t + x_{dt},\\
i_t &= r_t + E_t \pi_{t+1},\\
\pi_t &= \beta E_t \pi_{t+1} + \gamma y_t + x_{\pi t},\\
i_t &= \phi_{\pi,0}\pi_t + \phi_{\pi,1} E_t \pi_{t+1} + x_{it}
\end{aligned}
$$

when each of the disturbances x follows an AR(1). That solution is

$$
\begin{bmatrix} y_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 1-\rho_d\beta & \sigma(\rho_\pi(1-\phi_{\pi,1})-\phi_{\pi,0}) & -\sigma(1-\rho_i\beta) \\ \gamma & (1-\rho_\pi) & -\sigma\gamma \\ \gamma(\phi_{\pi,0}+\rho_d\phi_{\pi,1}) & (1-\rho_\pi)(\phi_{\pi,0}+\rho_\pi\phi_{\pi,1}) & (1-\rho_i)(1-\rho_i\beta)-\sigma\gamma\rho_i \end{bmatrix} \begin{bmatrix} z_{dt} \\ z_{\pi t} \\ z_{it} \end{bmatrix}, \tag{B17}
$$

where the z variables are scaled versions of the disturbances:

$$
\begin{bmatrix} z_{dt} \\ z_{\pi t} \\ z_{it} \end{bmatrix} = \begin{bmatrix} \rho_d & 0 & 0 \\ 0 & \rho_\pi & 0 \\ 0 & 0 & \rho_i \end{bmatrix} \begin{bmatrix} z_{dt-1} \\ z_{\pi t-1} \\ z_{it-1} \end{bmatrix} + \begin{bmatrix} v_{dt} \\ v_{\pi t} \\ v_{it} \end{bmatrix},
$$

$$
\begin{aligned}
x_{dt} &= [(1-\rho_d)(1-\rho_d\beta) + \sigma\gamma(\phi_{\pi,0} + \rho_d(\phi_{\pi,1}-1))]z_{dt},\\
x_{\pi t} &= [(1-\rho_\pi)(1-\rho_\pi\beta) + \sigma\gamma(\phi_{\pi,0} + \rho_\pi(\phi_{\pi,1}-1))]z_{\pi t},\\
x_{it} &= [(1-\rho_i)(1-\rho_i\beta) + \sigma\gamma(\phi_{\pi,0} + \rho_i(\phi_{\pi,1}-1))]z_{it}.
\end{aligned}
$$

These dynamics give us the impulse-response function to shocks. Looking at the right-most column of (B17), we see again that the Taylor rule coefficients ϕ do not appear in the response to the monetary policy shock z_{it}. The ϕ coefficients do appear in the responses to the other shocks z_{dt} and $z_{\pi t}$, however, which suggests a possibility to identify these parameters. In essence, responses to other shocks allow you to see some movement in inflation (and output) and the Fed's response to that movement without any intervening monetary policy shock. They offer the promise to gain the same identification of the 100 percent R^2 models without a monetary policy shock, but without making that unpalatable assumption. Equivalently, the other shocks (or combinations of endogenous variables that depend only on those shocks) seem to offer instruments.

Alas, this approach hinges on assumptions about the orthogonality of shocks. To identify a movement in z_{dt} (say) with no movement in z_{it}, we need to make assumptions about the correlation structure of the v shocks. As in the "stochastic intercept" discussion, we really do not have any such information. It also hinges on the lag-length restriction that shocks follow an AR(1).

C. Frictionless Model Extensions

1. Identification in All Equilibria of the Simple Model

The model of Section II has multiple equilibria. I studied identification for the new-Keynesian equilibrium choice. Here, I study identification in the other equilibria, that is, other choices of δ_{t+1}. Perhaps other equilibria do allow identification? Perhaps when $\phi < 1$ we can identify ϕ?

The answer is, other equilibria allow us to identify the pair ρ, ϕ, but we cannot identify ϕ separately from ρ. This result holds for all choices of δ_{t+1}, and for any value of ϕ, as long as only $\{\pi_t, i_t\}$ are observable. The new-Keynesian equilibrium is a special case in that only ρ is identified.

The equilibrium conditions are

$$E_t\pi_{t+1} = \phi\pi_t + x_t, \tag{B18}$$

$$x_t = \rho x_{t-1} + \varepsilon_t, \tag{B19}$$

$$i_t = \phi\pi_t + x_t. \tag{B20}$$

The equilibrium process for observables is

$$(i_{t+1} - \rho\pi_{t+1}) = \phi(i_t - \rho\pi_t) + \delta_{t+1}, \tag{B21}$$

$$(i_{t+1} - \phi\pi_{t+1}) = \rho(i_t - \phi\pi_t) + \varepsilon_{t+1}. \tag{B22}$$

(This is a different δ_t than in the text, though δ still indexes equilibria.) The new-Keynesian equilibrium is the case $i_t = \rho\pi_t$ and $\delta_t = 0$. Except for this special case in which the first equation is always zero, ρ and ϕ appear symmetrically in the equilibrium conditions. Therefore, we can identify the pair ρ, ϕ, but we cannot identify which is ρ and which is ϕ.

If $\phi > 1$, then we can in fact identify ρ and ϕ from the prior that $\phi > 1$ and $\rho < 1$. If the system does explode, then we can measure the speed of that explosion. If $\phi < 1$, then we will see both roots less than one and we cannot distinguish ϕ from ρ.

Equation (B22) follows by substituting the Taylor rule (B20) into the AR(1) process for x_t, (B19). To derive (B21), write

$$\begin{aligned} E_t(i_{t+1} - \rho\pi_{t+1}) &= E_t[(\phi - \rho)\pi_{t+1} + x_{t+1}] = (\phi - \rho)(\phi\pi_t + x_t) + \rho x_t \\ &= \phi(\phi\pi_t + x_t - \rho\pi_t) = \phi(i_t - \rho\pi_t). \end{aligned}$$

Then define δ_{t+1} as the unexpected component.

We can also write (B21) and (B22) as

$$i_{t+1} = \rho i_t + \phi(i_t - \rho\pi_t) - \frac{\phi}{\rho - \phi}\delta_{t+1} + \frac{\rho}{\rho - \phi}\varepsilon_{t+1},$$

$$\pi_{t+1} = \rho\pi_t + (i_t - \rho\pi_t) - \frac{1}{\rho - \phi}\delta_{t+1} + \frac{1}{\rho - \phi}\varepsilon_{t+1}.$$

Here we see in the new-Keynesian equilibrium choice $i_t = \rho\pi_t$ that only ρ remains identified since the second term becomes zero.

If we write the same dynamics in conventional VAR form,

$$i_{t+1} = -\phi\rho\pi_t + (\rho + \phi)i_t - \frac{\phi}{\rho - \phi}\delta_{t+1} + \frac{\rho}{\rho - \phi}\varepsilon_{t+1},$$

$$\pi_{t+1} = i_t - \frac{1}{\rho - \phi}\delta_{t+1} + \frac{1}{\rho - \phi}\varepsilon_{t+1},$$

we see that the interest rate equation has all the information in the VAR. If we infer ϕ and ρ from the interest rate equation of the VAR, we obtain a quadratic that treats ρ and ϕ as two roots, and we cannot tell which is which.

2. Impulse-Response Functions

It is useful to examine the message of the simple model in the language of impulse-response functions, since we often think about model dynamics and predictions in that framework. Again, the model is

$$i_t = r + E_t\pi_{t+1},$$

$$i_t = r + \phi\pi_t + x_t,$$

$$x_{t+1} = \rho x_t + \varepsilon_{t+1}.$$

The equilibrium condition is

$$\pi_{t+1} = \phi\pi_t + x_t + \delta_{t+1}.$$

The new-Keynesian solution, equations (6) and (7),

$$\pi_t = -\frac{x_t}{\phi - \rho}; \quad \delta_{t+1} = -\frac{\varepsilon_{t+1}}{\phi - \rho} \tag{B23}$$

gives us the impulse-response function of inflation to a monetary policy shock. Figure B1 plots the response of x_t, i_t, and π_t to the monetary policy shock.

Suppose that there is a positive monetary policy shock x_t in the Taylor rule, $i_t = r + \phi\pi_t + x_t$. Equation (B23) predicts that inflation jumps down immediately and then slowly recovers as the shock dynamics $x_{t+1} = \rho x_t + \varepsilon_{t+1}$ play out. This seems reasonable at first glance: monetary tightening lowers inflation. On second glance, this response seems completely counterintuitive in the context of this model. Real rates are constant, so the standard old-Keynesian intuition—higher nominal rates mean higher real rates, higher real rates lower demand, demand leads to less inflation—cannot possibly apply. The only way the Fed can possibly raise nominal rates in this model is to raise expected inflation. How can raising expected inflation lead to a sudden and persistent decline in actual inflation? In addition, notice that actual interest rates decline also throughout the episode. To an observer, inflation and interest rates both spontaneously move downward. The sense of a "tightening" is only that interest rates moved down less than ϕ times the downward movement in inflation.

How can this happen? The answer is, inflation "jumps" ($\delta_1 = -1/[\phi - \rho]$) to a new lower-inflation equilibrium in response to the monetary policy shock. Actual interest rates decline because at each date the disturbance x_t in the Taylor rule $i_t = r + \phi\pi_t + x_t$ is positive, but inflation π_t has jumped down by so much that actual interest rates are lower. An observer would never see a rise in interest rates in this tightening. The observer would see a decline only in interest rates, coincident with the large decline in inflation. The "tightening" comes because interest rates do not decline as much as inflation, so if the observer knew $\phi = 1.5$, he could infer a positive shock.

There is no economic force, no supply greater than demand, that forces the δ_1 jump in inflation in response to a shock. Any other value of δ_1 and π_1 would correspond to an equilibrium. But all those other equilibria are explosive, as shown. Inflation could even not change at all, $\pi_1 = 0$. Then we would see an increase in interest rates i_t—a standard tightening—followed by higher subsequent inflation, which is what you might have expected from a nominal interest rate rise in a frictionless economy. The response functions consist of jumps from one equilibrium to another, following the rule that we select locally bounded equilibria. That is how the model can achieve apparent magic: a positive interest rate shock lowers inflation in a completely frictionless model. It does not really "lower inflation"; it provokes the economy to jump to a different one of many equilibria, which has lower inflation.

Now, the jump $\delta_1 = -1/(\phi - \rho)$ means that the Treasury will raise taxes or lower spending just enough to change the present value of future surpluses as required by (22). From a fiscal point of view, we can regard (21) as fundamentally determining the price level; but with the Ricardian agreement of a passive fiscal policy, the whole point of the Taylor rule is just to induce the Treasury to embark on the contractionary fiscal policy that generates the required δ_1. From a fiscal point of view, the new-Keynesian response combines two shocks, x_t, the monetary policy shock, and δ_t, a shock to the present value of future surpluses.

The alternative "non-Ricardian" view suggests that we calculate a different response function: what happens if there is a monetary policy shock x_t but no fiscal response, so $\delta_t = 0$? Figure B2 presents a calculation. While we can pair "active" fiscal policy with $\phi > 1$, doing so leads to explosive solutions, so I change parameters to $\phi = 0.8$ in this example. Now, the policy shock $x_t = 1$ produces a 1 percent rise in interest rates and no change in inflation. Expected inflation and nominal rates are perfectly under the Fed's control with no fiscal response needed, so Taylor rule dynamics $i_t = r + \phi\pi_t + x_t$ now kick in and both interest rates and inflation take a long hump-shaped excursion. The monetary policy shock produces an increase in inflation, but that is what one might expect in a frictionless model.

Now, the new-Keynesian view illustrated by figure B1 and the non-Ricardian view illustrated by figure B2 are in fact observationally equivalent. In particular, the change in fiscal policy $\delta_1 = -1/(\phi - \rho)$ of figure B1 could have happened just by chance along with the change in monetary policy. The correlation between monetary and fiscal shocks could be exactly what the new-Keynesian model suggests they must be. One has to add identification assumptions to try to test the two models, which is beyond the scope of this paper. But the two models certainly suggest different responses to policy interventions! Any impulse-response function calculated with the assumption that δ_{t+1} will change in a specific way to produce a particular value of inflation at π_{t+1} will be quite different from an impulse-response function calculated assuming no fiscal cooperation, and thus $\delta_{t+1} = 0$. The differences between the two responses in three-equation models are equally stark.

D. Regressions in Model Data

In the simple model of Section II, we saw that if we ran the Taylor rule regression in data generated by the new-Keynesian model, we would recover the shock autocorrelation process, not the Taylor rule parameter. What happens if you run Taylor rule regressions in artificial data from real new-Keynesian models, such as the standard three-equation model? The answer is, in general, that the regressions do not recover the structural ϕ parameters. Even if the real data are drawn from the new-Keynesian model, the regressions do not measure its structural parameters.

For an example, once again I turn to Clarida et al. (2000) since their paper is so influential and since they include a model in the same paper as an estimate: if you run the regressions of the first half of Clarida et al.'s paper on artificial data from the model in the second half of their paper, you do not recover the Taylor rule parameters of that paper. (Jensen [2002] also estimates Taylor rules in artificial data from new-Keynesian models, finding estimated coefficients far from the true ones and often below one.)

1. Regressions in Real Data

As a backdrop, I replicate, update, and slightly extend a simple version of Clarida et al.'s Taylor rule regressions in table B1. I run regressions

$$i_t = a + \rho i_{t-1} + (1 - \rho)[\phi_{\pi,0}\pi_t + \phi_{\pi,1}E_t\pi_{t+1} + \phi_y(y_t - \bar{y}_t)] + \varepsilon_t$$

in quarterly data. I instrument expected inflation $E_t\pi_{t+1}$ on the right-hand side with current inflation.

Row 1 gives the basic result: a very high 3.66 coefficient on expected inflation. In the earlier period, row 5, this coefficient is only 0.85. Rows 3 and 7 show that the use of expected future inflation is a refinement, and shows a stronger result, but even a simple rule using only current inflation produces a 2.13 coefficient up from 0.78 in the earlier period. These coefficients are quite similar to Clarida et al.'s coefficients, which range from a baseline 2.15 (table 4) to as much as 3.13 (table 5). Coefficients of this magnitude are typical of the literature.

Row 2 and 6 present the raw regression coefficients, that is, not divided by $1 - \rho$. Much of the large ϕ_π estimate comes from a rise in the persistence ρ from 0.68 to 0.90, which implies larger long-run multipliers ϕ_π

from the same raw coefficients. Rows 4 and 8 remind us that these coefficients require a dynamic model (i_{t-1}) and computation of the long-run multiplier $\phi = b/(1-\rho)$. If you just run a Taylor rule in levels, you recover the fact that long-run levels of inflation and interest rates must move essentially one-for-one.

2. *Regressions in Model Data*

Now, suppose that data are generated from a new-Keynesian model. Will regressions such as the above recover the structural coefficients? Alas, even the simple three-equation model is complex enough that the answers, though computable, are algebraically large and hence not that enlightening. Therefore, I report a numerical investigation. Start with the most straightforward three-equation model,

$$y_t = E_t y_{t+1} - \sigma(i_t - E_t\pi_{t+1}) + x_{dt}, \tag{B24}$$

$$\pi_t = \beta E_t \pi_{t+1} + \gamma y_t + x_{\pi t},$$

$$i_t = \phi_{\pi 0}\pi_t + x_{it},$$

$$x_{jt} = \rho_j x_{jt-1} + \varepsilon_{jt}; \quad j = d,\ \pi,\ i. \tag{B25}$$

Here I have added shocks to each equation to avoid perfect correlations between variables.[4] Following Clarida et al., I use $\beta = 0.99$, $\sigma = 1.0$, and $\gamma = 0.3$.

I assume that all the shocks are independent of each other and have a common 0.9 autoregression coefficient. I solve the model, simulate a very long series of artificial data, and run Taylor rule regressions in the resulting artificial data. Table B2 collects results.

The rows of table B2 labeled "model" give the assumed value of policy parameters ρ and ϕ. We want to see if regressions can recover these values.

In row 1, I simulate data with no monetary policy shock, $\sigma_i = 0$. This regression does recover the true policy parameter. It must. With no stochastic intercept and no error term, $i_t = \phi_{\pi,0}\pi_t$; if there is any variation in inflation at all, we must recover the true $\phi_{\pi 0}$. The trick and cost are easy to see: the R^2 is 100 percent. Thus, we could easily reject this model in any real data set.

Row 2 adds back a monetary policy shock with 1 percent standard deviation, to remove the 100 percent R^2 prediction. Now, as in the simple model, the right-hand variable is correlated with the error and we recover a coefficient of 1.44, not the true coefficient of 2.00. In row 3 we see that if there is only a monetary policy shock (i.e., if the variance of the monetary policy shock is much larger than that of the other shocks), the estimated coefficient declines to 0.86. Row 4 shows what happens if we estimate an interest rate persistence parameter ρ where there is none in the underlying model. Interestingly, we recover a rather large spurious persistence $\rho = 0.42$ estimate along with the inconsistent $\phi_{\pi,0}$ estimate.

The second group of results in table B2 add an output response and persistence in the Taylor rule. The picture is the same, with larger inconsistencies. The estimated persistence parameter ρ is lower, 0.65–0.55, than the true value $\rho = 0.90$. In part as a result, the estimated $\phi_{\pi,0}$ are 0.86–0.78, much lower than the true $\phi_{\pi,0} = 2$, and the ϕ_y estimate is destroyed to values near zero. The estimated values are in the zone of multiple local equilibria. If

[4] Clarida et al. specify a slightly different model, their (6)–(9), p. 169, and in my notation:

$$y_t = E_t y_{t+1} - \sigma(i_t - E_t\pi_{t+1}) + x_{dt},$$

$$\pi_t = \beta E_t \pi_{t+1} + \gamma(y_t - \bar{y}_t),$$

$$i_t = \rho i_{t-1} + (1-\rho)[\phi_{\pi 1} E_t \pi_{t+1} + \phi_y (y_t - \bar{y}_t)],$$

$$x_d = \rho_d x_{dt-1} + \varepsilon_{dt}; \quad \bar{y}_t = \rho_{\bar{y}} \bar{y}_{t-1} + \varepsilon_{\bar{y}t}.$$

Potential output $\bar{y}_t$ varies, there is no additional shock to the Phillips curve, and there is no monetary policy shock. I had hoped to calculate "what happens if you run Clarida et al.'s regressions in data from Clarida et al.'s model," but unfortunately this is impossible. This model produces a perfect correlation between π_t, $E_t\pi_{t+1}$ and $(y_t - \bar{y}_t)$. Hence the right-hand variables of the Taylor rule regression are perfectly correlated; one cannot run Clarida et al.'s regressions in artificial data from their model. If one adds errors to the Phillips curve or monetary policy equation, one can break that correlation, but then it is no longer really their model, and no lessons beyond what I find in the simpler model covered in the text emerge. The point of course is not to deconstruct one paper but to understand a whole literature using that paper as an example, so the model in the text that has a better chance of success is appropriate.

you run Taylor rule regressions in data generated from this model, you have no hope of recovering the true policy function.

Regressions in model data not only do not capture the true coefficients, they also produce far higher R^2's than in actual data, which is a revealing failure. If the model is right, regressions in artificial data should produce all features of the estimated regression, including its R^2. Yet even with all three shocks, R^2 is almost exactly one.

The R^2 of a highly serially correlated variable is misleading since so much explanatory power comes from the lagged dependent variable. Hence, I also compute in tables B1 and B2 the R^2 of the component of interest rates not predicted by the lagged interest rate, $R^2(i_t - \rho i_{t-1})$, which is $1 - \sigma^2(\varepsilon)/\sigma^2(i_t - \hat{\rho} i_{t-1})$. These R^2's in data generated by the model are between 0.92 and 1, much higher than the values from actual data of 0.2–0.3, shown in table B1.

The prediction of very high R^2's in artificial data seems hard to avoid. The reason for the high R^2's is that the right-hand variables all jump when there is a shock. Thus, even if we have a very large policy shock, the right-hand variables respond and incorporate its information. I read this as an indication that right-hand variables in the real world are not jumping as predicted by the model.

The estimates in rows 8 and 9 of table B2 assume and estimate a rule with expected future inflation, using instrumental variables. Now we do not recover the true values, even when there are no policy shocks so that $R^2 = 1.00$. Mechanically, the most important instrument for expected inflation is past inflation, with a first-stage coefficient of about 0.5. Thus, the two-stage least-squares estimate is roughly double the OLS estimate. In addition, this estimate is much higher than the true value, a counterexample to the impression one might otherwise get that estimated coefficients are always lower than true ones. Once again, adding a monetary policy shock dramatically lowers the estimate, without substantially changing the prediction of a very high R^2.

Finally, row 10 investigates the stochastic intercept. Here, I assume that the policy rule is

$$i_t = \rho i_{t-1} + \phi_\pi \pi_t + \phi_y (y_t - \bar{y}_t),$$

that is, no error term, and policy responds to the output gap rather than output. In the model, I also write the Phillips curve as a function of the output gap,

$$\pi_t = \beta E_t \pi_{t+1} + \gamma (y_t - \bar{y}_t) + x_{\pi t}.$$

Now, if we can observe potential output and estimate a policy rule with that value, we can recover the structural parameters if we swallow the usual 100 percent R^2. However, the Fed typically has much more information about "potential output" than we do, and its assessment of potential is a prime source of disturbances to estimated monetary policy equations. (Of course, potential output is likely to be correlated to other shocks as well, but I ignore this fact for the calculation.) The Fed's potential output guess is a source of a stochastic intercept. Thus, what happens when we estimate a policy rule using only actual output? Row 10 shows the answer, again showing that we recover nothing like the true policy parameters, even without any monetary policy shock.

3. Large Estimates—a New-Keynesian Interpretation

How do estimated Taylor rules recover large coefficients, and coefficients that change around 1980? Really, the answer is "we don't care." Once we know that the coefficients are mongrels, mixing irrelevant model parameters and shock dynamics, who cares how they turn out? They are not measuring anything important in the context of the new-Keynesian model.

But many of the examples so far have all shown a downward bias. Is the fact that estimated Taylor rules show large coefficients embarrassing? Again, the answer is no. It is easy to give plausible examples in which the estimated Taylor rules give much larger than actual coefficients, and their coefficients change from below to above one depending on small other changes in specification.

For example, introduce an IS or real interest rate shock z_t to the simple model of Section II, so the model is

$$i_t = r + E_t \pi_{t+1} + z_t; \quad z_t = \rho_z z_{t-1} + \varepsilon_{zt};$$

$$i_t = r + \phi \pi_t + x_t; \quad x_t = \rho_x x_{t-1} + \varepsilon_{xt}.$$

Now the equilibrium is

$$E_t \pi_{t+1} = \phi \pi_t + x_t - z_t$$

and the forward-looking solutions are

$$\pi_t = -\frac{x_t}{\phi - \rho_x} + \frac{z_t}{\phi - \rho_z},$$

$$i_t = -\frac{\rho_x x_t}{\phi - \rho_x} + \frac{\phi z_t}{\phi - \rho_z}.$$

The x shock lowers inflation, so inflation and x are negatively correlated, leading to downward bias in a Taylor rule estimate. A z shock raises inflation, however, so if z and x are positively correlated, we can have inflation positively correlated with the monetary disturbance x and a positive bias in the Taylor rule estimate.

This is also a sensible specification. Left alone, the IS shocks z_t would induce inflation variability. A central bank that wants to minimize the variance of inflation would deliberately introduce a policy disturbance $x_t = [(\phi - \rho_x)/(\phi - \rho z)]z_t$ in order to offset the IS shocks. This is an example of a "stochastic intercept," a "Wicksellian policy" in which the monetary policy disturbance offsets other structural shocks.

The bias can be large, and it is largest when the central bank is close to doing its job of offsetting z shocks. For example, let $\rho_x = \rho_z = \rho$, and let z and x be perfectly positively correlated. Then the regression coefficient of i_t on π_t is[5]

$$\hat{\phi} = \frac{\mathrm{Cov}\,(i_t,\ \pi_t)}{\mathrm{Var}\,(\pi_t)} = \frac{\rho - \phi(\sigma_z/\sigma_x)}{1 - (\sigma_z/\sigma_x)}.$$

A parameter σ_z just slightly higher than σ_x gives large positive coefficients. For example, $\rho = 0.7$, $\phi = 1.1$, and $\sigma_z/\sigma_x = 1.2$ produces an estimated coefficient similar to the estimated values,

$$\hat{\phi} = \frac{0.7 - 1.1 \times 1.2}{1 - 1.2} = 3.1. \tag{B26}$$

However, if the IS shocks z_t and monetary policy shocks x_t are uncorrelated, the estimated coefficient is instead between ρ and ϕ:

$$\hat{\phi} = \frac{\rho + \phi(\sigma_z^2/\sigma_x^2)}{1 + (\sigma_z^2/\sigma_x^2)}.$$

For example, then, $\rho = 0.7$, $\phi = 1.1$, $\sigma_z/\sigma_x = 1$ gives

$$\hat{\phi} = \frac{0.7 + 1.1}{2} = 0.9. \tag{B27}$$

Thus, one possibility consistent with standard estimates is this: a new-Keynesian model with constant Taylor rule $\phi = 1.1$ operated throughout the period. Before 1980, the Fed was not very good at offsetting IS shocks, so we estimate $\hat{\phi} = 0.9$ as in (B27). After 1980, the Fed got much better at offsetting IS shocks, and we estimate $\hat{\phi} = 3.1$. Inflation volatility declined because the Fed got better at offsetting IS shocks, not because it changed the Taylor rule parameter.

I do not argue that this is what happened. The point of the determinacy section of this paper is that the $\phi > 1$ passive-fiscal solutions are fundamentally flawed. It is simply one logical possibility and a way to remind ourselves that once estimated coefficients do not measure structural parameters, changes in those mismeasured coefficients can reflect all sorts of changes in model structure.

Further pursuit of an "explanation for Taylor rule results" is just not interesting. Producing spurious $\hat{\phi}$ estimates consistent with the empirical findings is a necessary condition for the right model but not sufficient. Many wrong models will also produce the observed $\hat{\phi}$ estimates. More important, producing this mongrel

[5] Algebra:

$$\frac{\mathrm{Cov}\,(i_t,\ \pi_t)}{\mathrm{Var}\,(\pi_t)} = \frac{\mathrm{Cov}\left(-\frac{\rho x_t}{\phi-\rho} + \frac{\phi z_t}{\phi-\rho},\ -\frac{x_t}{\phi-\rho} + \frac{z_t}{\phi-\rho}\right)}{\mathrm{Var}\left(-\frac{x_t}{\phi-\rho} + \frac{z_t}{\phi-\rho}\right)} = \frac{\rho\sigma_x^2 + \phi\sigma_z^2 - (\rho+\phi)\sigma_x\sigma_z}{\sigma_x^2 + \sigma_z^2 - 2\sigma_x\sigma_z},$$

which simplifies to the given expression.

coefficient is not likely to be much help in discriminating between models focused on other effects. We do not learn much about new models by asking if they produce investment equation regressions, consumption function regressions, regressions of output on lags of money stocks, or any other now-spurious regression run in the past when thinking about other models.

E. Rules with Leads and Lags

It is often claimed that the principle "raise interest rates more than one-for-one with inflation" is robust to details of model and rule specification (Taylor 1999; Woodford 2003, among many others). In fact, determinacy is very sensitive to small changes in timing, whether the central bank reacts to current, lagged, or expected future inflation. This is true both in the simple model and in the standard three-equation model. We have already seen a hint of this result in Section A.2; a small change in timing there changed the crucial eigenvalue to infinity. This section also documents a natural question from the text: "what happens if we change the timing?" ("Robustness" may refer to model fit, impulse responses, and policy analysis given determinacy. I do not address these issues.)

1. The Frictionless Model

Start with our simple Fisher equation model (1) but allow the Fed to respond to expected future inflation rather than current inflation, and for simplicity I ignore the disturbance x_t. Determinacy means $\pi_t = 0$. Generalize the simple model to change the timing, that is,

$$i_t = r + E_t\pi_{t+1},$$

$$i_t = r + \phi E_t\pi_{t+j}.$$

Again, we find equilibria by eliminating i_t between these two equations.

For $j = 0$ (contemporaneous inflation), the equilibrium condition is

$$E_t\pi_{t+1} = \phi\pi_t;$$

as we have seen, the condition for a unique local equilibrium is $||\phi|| > 1$.

For $j = 1$, a reaction to expected future inflation, the equilibrium condition becomes

$$E_t\pi_{t+1} = \phi E_t\pi_{t+1}.$$

If $\phi = 1$, anything is a solution. For any $\phi \neq 1$ (both $\phi > 1$ and $\phi < 1$), solutions must obey

$$E_t\pi_{t+1} = 0; \quad \pi_t = \delta_{t+1}. \tag{B28}$$

We conclude that inflation must be white noise: real rates are constant. But that is all we can conclude. No value of ϕ gives even local determinacy.

For $j = 2$, we have

$$E_t\pi_{t+1} = \phi E_t\pi_{t+2}.$$

Now a necessary condition for "unstable" or "forward-looking" equilibrium is reversed, $||\phi|| < 1$. Since interest rates react to inflation two periods ahead and interest rates control expected inflation one period ahead, the interest rate and one-period-ahead inflation must move less than two-period-ahead inflation if we want an explosive root. And even this specification is now not enough to give us a unique local equilibrium since there is an E_t on both sides of the equation. So $\pi_{t+1} = \delta_{t+1}$, $E_t(\delta_{t+1}) = 0$ is a solution for any value of ϕ.

In sum, in this simple model, Taylor determinacy disappears as soon as the Fed reacts to expected future rather than current inflation, and the solution is extremely sensitive to the timing convention. All the dynamics of the model, which are crucial to the idea of using forward-looking solutions to determine expectational errors, rely entirely on the assumed dynamics by which the Fed reacts to inflation.

A value $j = -1$ gives really weird dynamics, but preview what will happen in continuous time. Now the equilibrium condition is

$$E_t\pi_{t+1} = \phi\pi_{t-1}.$$

Even and odd periods live in their own disconnected equilibria!

2. Continuous Time and Dynamics

The timing issue drives the apparently strange modifications one needs to make in order to take the continuous-time limit. Benhabib, Schmitt-Grohé, and Uribe (2001) present one such model. If we eliminate money from their perfect-foresight model, the Fisher equation is simply

$$i_t = r + \pi_t.$$

In continuous time (and with continuous sample paths) the distinction between past and expected future inflation vanishes. If we write a Taylor rule

$$i_t = \phi(\pi_t),$$

as they do, we see that this system behaves exactly as the $i_t = r + \phi E_t \pi_{t+1}$ or $j = 1$ case above: If $\phi(\pi) = r + \pi_t$, anything is an equilibrium; otherwise there is a unique equilibrium in perfect foresight, but the same multiplicity once we allow expectational errors. It seems that there are no dynamics (or the dynamics happen infinitely quickly), so the forward-looking trick to determine expectational errors disappears.

Benhabib et al. do have dynamics that look a lot like figure 1 (see their fig. 1). However, these dynamics come from an entirely different source. Their model has money in the utility function. The dynamics of inflation in their model come from the standard interest elasticity of money demand, much like Cagan (1956) hyperinflation dynamics under a money target.

Here is their argument in continuous time: With money in the utility function and a constant endowment, the first-order condition for money M_t versus consumption C_t implies a "money demand" curve (my notation) $M_t/P_t = L(Y, i_t)$. Thus, we can write the marginal utility of consumption in equilibrium as

$$u_c(C_t, M_t/P_t) = u_c[Y, L(Y, i_t)] = \lambda(i_t) = \lambda[\phi(\pi_t)],$$

where the last equalities define the function λ. Differentiating and using the continuous-time first-order condition

$$\frac{\dot{u}_c}{u_c} = i - \pi - \rho,$$

where ρ is the rate of time preference, we have

$$\dot{\pi}_t = \frac{\lambda[\phi(\pi_t)][\phi(\pi_t) - \pi_t - \rho]}{\lambda'[\phi(\pi_t)]\phi'(\pi_t)}.$$

This differential equation in π_t turns out to look just like figure 1.

The idea may be clearer in the discrete-time formulation. With a Taylor rule $i_t = \phi(\Pi_{t+1})$, equation (25) becomes

$$\Pi_{t+1} = \beta[1 + \phi(\Pi_{t+1})]\frac{u_c(Y, M_{t+1}/P_{t+1})}{u_c(Y, M_t/P_t)}.$$

If we had no money in the utility function, you can see how once again we are stuck. There are no dynamics. However, the interest elasticity of money demand offers hope. Substituting for money demand, we have

$$\Pi_{t+1} = \beta[1 + \phi(\Pi_{t+1})]\frac{u_c[Y, L(Y, i_{t+1})]}{u_c[Y, L(Y, i_t)]},$$

$$\Pi_{t+1} = \beta[1 + \phi(\Pi_{t+1})]\frac{u_c\{Y, L[Y, \phi(\Pi_{t+2})]\}}{u_c\{Y, L[Y, \phi(\Pi_{t+1})]\}}.$$

Now we again have a difference equation that can look like figure 1.

In sum, despite the behavior of Benhabib, Schmitt-Grohé, and Uribe's (2002) model superficially similar to the frictionless models studied above, we see they are fundamentally different. Their model cannot work in a frictionless economy; it relies on the dynamics induced by interest elastic money demand rather than dynamics induced by the policy rule.

To mirror the sort of dynamics we have seen from $i_t = \phi(\pi_t)$ rules in continuous time, one must specify some explicit time lag between inflation and the interest rate, $i_t = \phi(\pi_{t-k})$, or $i_t = \int_{k=0}^{\infty} f(k)\pi(t-k)dk$. For example, Sims (2003) models a Taylor rule in continuous time in this way as

$$\frac{di}{dt} = \theta_0 + \theta_1 \frac{1}{P}\frac{dP}{dt} - \theta_2 i. \tag{B29}$$

Here, more inflation causes the Fed to raise the rate of change of interest rates. Sims also has a Fisher equation

$$i = \rho + \frac{1}{P}\frac{dP}{dt}.$$

Sims solves for the interest rate

$$\frac{di}{dt} = \theta_0 + (\theta_1 - \theta_2)i_t - \theta_1\rho,$$

thus wanting $\theta_1 < \theta_2$ for forward-looking solutions. The specification (B29) is not an ad hoc peculiarity; it is exactly the sort of modification we must make for Taylor rule dynamics to work in continuous time.

3. *Timing in the Three-Equation Model*

One might think that sensitivity to timing is a peculiarity of the frictionless model, that price stickiness will smooth things out in some sense. In particular, the forward-looking rule $i_t = r + \phi E_t\pi_{t+1}$ seems to make much more sense in the three-equation model because it ensures that real rates $r_t = i_t - E_t\pi_{t+1}$ will rise after an increase in inflation. In the frictionless model, a rise in real rates is impossible, which could account for that model falling apart when the Taylor rule becomes forward looking. However, that could be old-Keynesian intuition sneaking in. The point of the Taylor rule is not to raise real rates, lower demand, and lower inflation; the point of the Taylor rule in a new-Keynesian model is to destabilize the system so that it explodes for all but one initial value. Let's see.

The familiar three-equation model is also sensitive to timing. The Taylor rule parameter regions required to produce a forward-looking solution vary considerably whether the central bank reacts to current or expected future inflation and whether the central bank reacts to output. The equations are not that revealing: since we are studying roots of quadratic equations, there are multiple special cases for real roots, imaginary roots, and roots equal to one or to negative one, and one must check the smaller of two roots. Therefore, I focus here on a graphical analysis of some special cases.

Figure B3 presents the simplest case, the region of determinacy for a Taylor rule $i_t = \phi_{\pi,0}\pi_t$ in the standard three-equation model.

The conventional determinacy region $\phi_{\pi,0} > 1$ is visible. A region of large negative $\phi_{\pi,0}$ with $\sigma\gamma > 0$ is also visible, corresponding to the $\phi < -1$ region studied in the simple model of Section II. Finally, eigenvalues are a property of the whole model, not just the interest rate rule. Here, if $\sigma\gamma < 0$, all sorts of interesting ϕ regions lead to determinacy or not, including a region with $\phi_{\pi,0} > 1$ that is nonetheless indeterminate and one with $||\phi_{\pi,0}|| < 1$ that is nonetheless determinate. Of course $\sigma > 0$ and $\gamma > 0$ are conventional parameter restrictions, but this simple example alerts us that in more complex models eigenvalues are likely to depend on all parameters, not just the intuitively appealing (using old-Keynesian intuition) Taylor rule parameters.

Figure B4 presents the regions of local determinacy in the three-equation model for a policy rule that responds to expected future inflation $i_t = \phi_{\pi,1}E_t\pi_{t+1}$—again, perhaps the most interesting case. In the usual parameter region $\sigma\gamma > 0$, we see a comforting region $\phi_{\pi,1} > 1$. The rest of the parameter space is quite different from the case $i_t = \phi_{\pi,0}\pi_t$ of figure B3, however. In particular, as King (2000, fig. 3b) notices, there is a second region of large $\phi_{\pi,1}$ that again leads to local indeterminacy. In this region, both eigenvalues are negative, but one is less than one in absolute value. "Sunspots" that combine movements in output and inflation, essentially offsetting in the Phillips curve so that expected inflation does not move much, lead to stable dynamics.

Figure B4 plots regions of local determinacy combining responses to current and future inflation $\phi_{\pi,0}$ and $\phi_{\pi,1}$. We see the expected condition $\phi_{\pi,0} + \phi_{\pi,1} > 1$ in the downward-sloping line of the left-hand part of the plot. However, this line disappears when $\phi_{\pi,0} < -(1-\beta)/\sigma\gamma = -0.025$. A greater $\phi_{\pi,1}$ cannot make up for an even slightly negative $\phi_{\pi,0}$. In fact, we see that the sum $\phi_{\pi,0} + \phi_{\pi,1}$ must be less than one for local determinacy when $\phi_{\pi,0} < 0$, an excellent counterexample to the view that $\phi_{\pi,0} + \phi_{\pi,1} > 1$ is a robust result. In both cases we see again that $\phi_{\pi,1}$ cannot get too big or again we lose determinacy for any value of $\phi_{\pi,0}$ (see fig. B5).

King concludes from experiments such as these that "forward-looking rules, then, suggest a very different pattern of restrictions are necessary to assure that there is a neutral level of output." He also takes seriously the regions with local indeterminacy (one eigenvalue less than one) despite large $\phi_{\pi,1}$ as important policy advice, writing, "It is important, though, that it [monetary policy] not be too aggressive, since the figure shows that some larger values are also ruled out because these lead to indeterminacies" (80).

As a slightly novel example, consider what happens if the Fed responds to expected inflation two periods ahead as well as one period ahead; that is, consider a Taylor rule of the form

$$i_t = \phi_{\pi,0}\pi_t + \phi_{\pi,1}E_t\pi_{t+1} + \phi_{\pi,2}E_t\pi_{t+2}.$$

Figure B6 presents the region of determinacy (both roots greater than one in absolute value) for this case. As you can see, there is some sense in the view that $\phi_{\pi,0} + \phi_{\pi,1} + \phi_{\pi,2} > 1$ is important for determinacy. (As one raises $\phi_{\pi,0}$, the region of local determinacy descends as you would expect.) However, there is more to it than that. We must have $\phi_{\pi,2} \le 0$: the Fed must respond negatively if at all to expected inflation two periods out! Furthermore, we see another example in which too large $\phi_{\pi,1}$ leads to indeterminacy for any $\phi_{\pi,2}$.

Allowing responses to output adds a whole interesting new range of possibilities. Figure B7 presents the determinacy region for rules limited to current output and inflation, $i_t = \phi_{\pi,0}\pi_t + \phi_{y,0}y_t$, for $\beta = 0.95$, $\sigma = 2$, $\gamma = 1$. Figure B8 presents the same region for rules limited to expected future output and inflation, $i_t = \phi_{\pi,0}E_t\pi_{t+1} + \phi_{y,0}E_ty_{t+1}$.

In figure B7, you can see that output responses can substitute for inflation responses. Rules are possible that use only output responses, ignoring inflation all together. Depending on ϕ_y and the other parameters of the model, almost any value of ϕ_π can be consistent or inconsistent with determinacy.

Figure B8 shows that the region of determinacy using expected *future* output and inflation is radically different from the one that uses *current* output and inflation. In particular, almost the whole range $\phi_y < 0$ is wiped out, and there are severe constraints on how strong the inflation and output responses can be.

Two-dimensional graphs can only do so much justice to this seven-dimensional space ($\phi_{\pi,0}$, $\phi_{\pi,1}$, $\phi_{y,0}$, $\phi_{y,1}$, β, γ, σ), of course. The determinacy $\|\lambda\| = 1$ boundaries in this case are as follows. (These conditions are derived below and included in the plots. To turn them into boundaries, one has to also check that the other eigenvalue is greater than one.)

1. $\sigma\phi_{y,1} \neq 1$, real roots, $\lambda = 1$:

$$(\phi_{\pi,0} + \phi_{\pi,1} - 1) + \frac{1-\beta}{\gamma}(\phi_{y,1} + \phi_{y,0}) = 0. \tag{B30}$$

2. $\sigma\phi_{y,1} \neq 1$, real roots, $\lambda = -1$:

$$(1 + \phi_{\pi,0} - \phi_{\pi,1}) - \frac{1+\beta}{\gamma}(\phi_{y,1} - \phi_{y,0}) = -2\frac{1+\beta}{\sigma\gamma}.$$

3. $\sigma\phi_{y,1} \neq 1$, complex roots:

$$\gamma\phi_{\pi,0} + \phi_{y,0} + \beta\phi_{y,1} = \frac{\beta-1}{\sigma}.$$

4. $\sigma\phi_{y,1} = 1$, $\lambda = 1$:

$$(\phi_{\pi,0} + \phi_{\pi,1}) + \frac{1-\beta}{\sigma\gamma}(1 + \sigma\phi_{y,0}) = 1.$$

5. $\sigma\phi_{y,1} = 1,\lambda = -1$:

$$\phi_{\pi,0} - \phi_{\pi,1} + \frac{1+\beta}{\sigma\gamma}(1 + \sigma\phi_{y,0}) = -1. \tag{B31}$$

Equations (B30)–(B31) show a variety of interesting additional interactions. Four of the five conditions do not involve $\phi_{\pi,0} + \phi_{\pi,1}$ and $\phi_{y,0} + \phi_{y,1}$; we see analytically that responding to current versus future output and inflation are not the same thing. In fact the second and last conditions include $\phi_{\pi,0} - \phi_{\pi,1}$: the two responses have opposite effects. (Unsurprisingly, these are conditions in which the eigenvalue is equal to negative one.) The other conditions trade off responses with coefficients that depend on structural parameters.

The point of all these examples is to emphasize that the Taylor rule does not "stabilize inflation" in new-Keynesian models; rather it threatens explosive equilibria. The examples emphasize that the regions of determinacy depend on the entire system, not just the policy rule. The regions are sensitive functions of policy rule parameters and timing, as well as economic parameters. The "robustness" may be a feature of old-Keynesian models and stories, but not of these models.

4. Zero-Inflation and Zero-Gap Equilibria

With the solution (B17) in hand we can easily check that $\pi_t = 0$ and $y_t = 0$ (there is no $\bar{y}$ here) are not achievable, so to achieve those equilibria a "stochastic intercept" policy is necessary. The Fed could set x_{it} and hence z_{it} to zero, eliminating this source of variance. Given that the variance of the z_t is not zero, there is nothing to do about any of the loadings of y_t or π_t except the loading of y_t on $z_{\pi t}$ (the top row, center column of [B17]). That loading can be set to zero with $\phi_{\pi,0} = \rho_\pi(1 - \phi_{\pi,1})$. But the other loadings remain. Hence, the only hope is to set to zero the variances of the z_t by altering the coefficients that relate z_t to x_t. To obtain this result, we need to explode the denominators,

$$\phi_{\pi,0} + (\phi_{\pi,1} - 1)\rho_d \to \infty,$$

$$\phi_{\pi,0} + (\phi_{\pi,1} - 1)\rho_\pi \to \infty.$$

The only way to do this is to send the individual ϕ_π responses to infinity. Doing so sets $\pi_t = 0$. The loading of y on z_π offsets this operation, though, so even setting $\phi_\pi = \infty$ does not give $\sigma(y) = 0$.

Table B1. Taylor Rule Regressions

	i_{t-1}	π_t	$E_t\pi_{t+1}$	$y_t - \bar{y}_t$	R^2	$R^2(i_t - \hat{\rho} i_{t-1})$
			A. 1984:1–2010:1			
1. $\phi = b/(1-\rho)$	.90		3.66	.64	.95	.23
	(.05)		(1.69)	(.35)		
2. b	.90		.36	.06		
	(.05)		(.14)	(.04)		
3. $\phi = b/(1-\rho)$	.92	2.13		1.00	.96	.19
	.02	(1.09)		(.33)		
4. b		.99		.48	.36	
		(.35)		(.08)		
			B. 1960:1–1979:2			
5. $\phi = b/(1-\rho)$	.68		.85	.54	.89	.49
	(.08)		(.13)	(.16)		
6. b	.68		.27	.17		
	(.08)		(.08)	(.03)		
7. $\phi = b/(1-\rho)$	.74	.78		.76	.89	.43
	.07	(.17)		(.21)		
8. b		.75		.28	.75	
		(.04)		(.08)		

Note.—i is the Federal funds rate, measured in the first month of the quarter, π is the GDP deflator, $y - \bar{y}$ is log GDP less log Congressional Budget Office potential GDP, b are the raw regression coefficients, $b/(1-\rho)$ divides the coefficients on π and $y - \bar{y}$ by $1 - \rho$ to give estimates of ϕ, and $R^2(i_t - \hat{\rho} i_{t-1})$ gives $1 - \sigma^2(\varepsilon)/\sigma^2(i_t - \hat{\rho} i_{t-1})$. In IV regressions I use π_t as an instrument for $E_t\pi_{t+1}$ and R^2 reports the variance of the instrumented right-hand side divided by the variance of the left-hand variable. Standard errors (in parentheses) by GMM include ρ estimation error and correct for heteroskedasticity. The standard errors in rows 4 and 8 are also corrected for serial correlation with a 12 quarter Newey-West weight.

Table B2. Regressions in Long ($T = 20{,}000$) Artificial Data from the Three-Equation New-Keynesian Model (B21)–(B22)

	σ_i	$\sigma_d = \sigma_\pi$	ρ	$\phi_{\pi,0}$	$\phi_{\pi,1}$	ϕ_y	R^2	$R^2(i_t - \hat{\rho} i_{t-1})$
Model			0	2		0		
1	0	1		2.00			1.00	
2	1	1		1.44			.85	
3	1	0		.86			1.00	
4	1	1	.42	1.55			.90	
Model			.90	2		1		
5	1	1	.65	.86		.03	.99	.92
6	1	.5	.59	.78		−.02	.99	.97
7	1	.01	.55	.76		−.03	1.00	1.00
Model			.90		3	1		
8. IV	0	1	.90		6.5	1.25	1.00	1.00
9. IV	1	1	.72		1.02	.01	.98	.87
Model			.90	2		1		
10	0	.5	.85	1.30		.28	.99	.90

Note.—The regression equation is $i_t = \rho i_{t-1} + (1-\rho)(\phi_{\pi,0}\pi_t + \phi_{\pi,1}E_t\pi_{t+1} + \phi_y y_t) + \varepsilon_t$. σ_i, σ_d, and σ_π give the standard deviation in percent of shocks. "Model" gives the true parameters; numbered rows are estimates. IV estimates use current inflation to instrument for expected inflation. In row 10 the policy rule and Phillips curve respond to $y_t - \bar{y}_t$, but the regression only uses only y_t. $R^2(i_t - \hat{\rho} i_{t-1})$ gives $1 - \sigma^2(\varepsilon)/\sigma^2(i_t - \hat{\rho} i_{t-1})$.

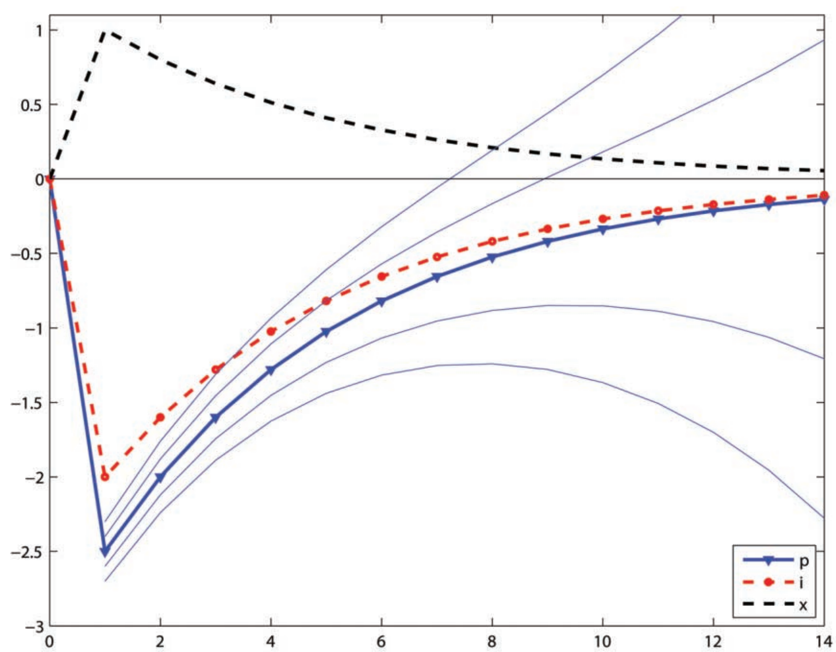

Fig. B1.—Response of the simple new-Keynesian model to a monetary policy shock. The model is $i_t = r + E_t\pi_{t+1}$; $i_t = \phi\pi_t + x_{t+1}$; $x_{t+1} = \rho x_t + \varepsilon_{t+1}$; the equilibrium is $\pi_t = -x_t/(\phi - \rho)$. Thin lines show explosive paths if δ_1 does not jump to the right value.

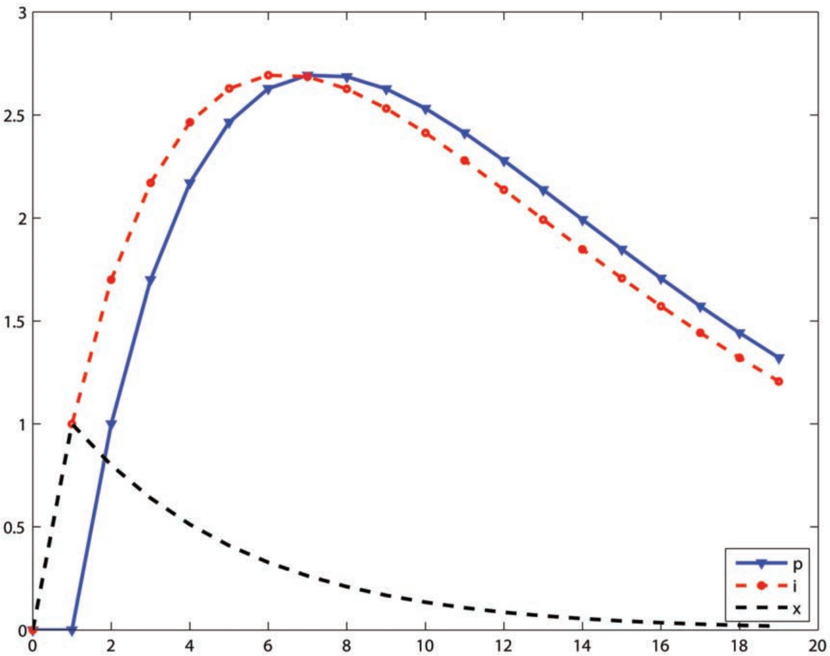

FIG. B2.—Impluse-response function of the simple model to a monetary policy shock, with no contemporaneous change in present value of surpluses, so $\delta_{t+1} = 0$.

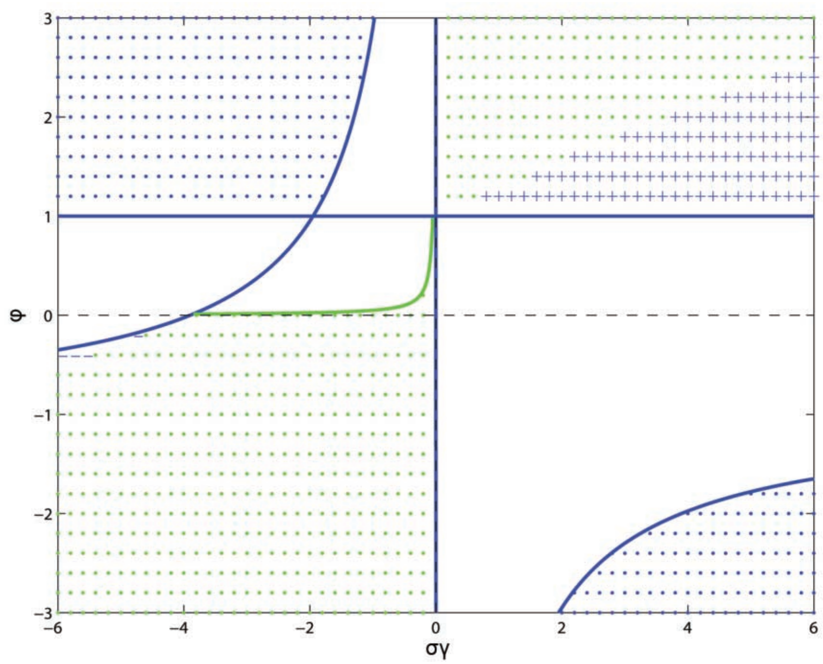

FIG. B3.—Regions of unique local equilibrium (both roots greater than one) for the three-equation new-Keynesian model, in which the Fed follows a Taylor rule $i_t = \phi\pi_t$, $\beta = 0.95$. Blue indicates real roots, green indicates complex roots; $+$, $-$, and $\cdot$ indicate positive, negative, and mixed roots, respectively. Lines denote regions in which one root is equal to one.

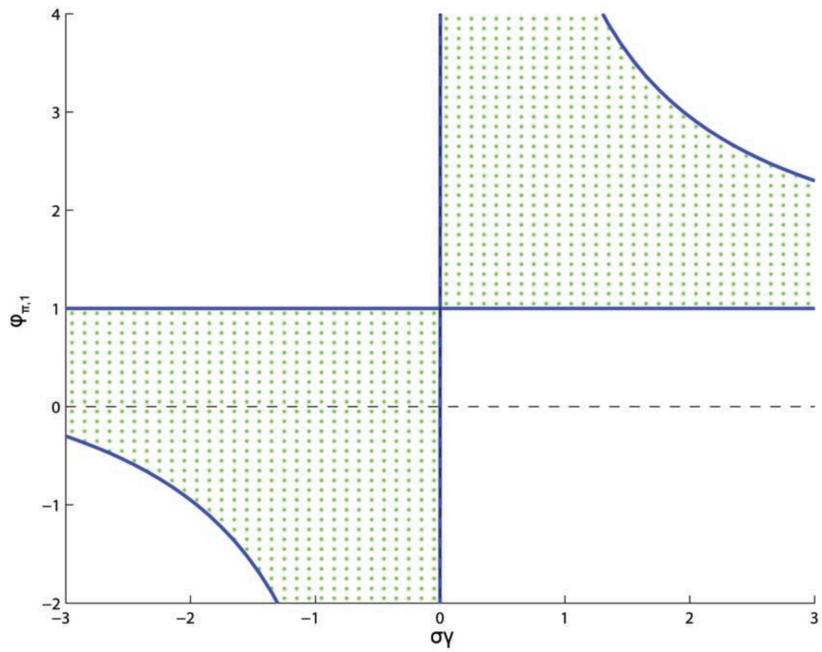

FIG. B4.—Zones of local determinacy—both eigenvalues greater than one in absolute value—when the policy rule responds to expected future inflation $i_t = \phi_{\pi,1} E_t \pi_{t+1}$. The plotted boundaries are $\phi_{\pi,1} = 1$, $\phi_{\pi,1} = 1 + 2(1+\beta)/(\sigma\gamma)$.

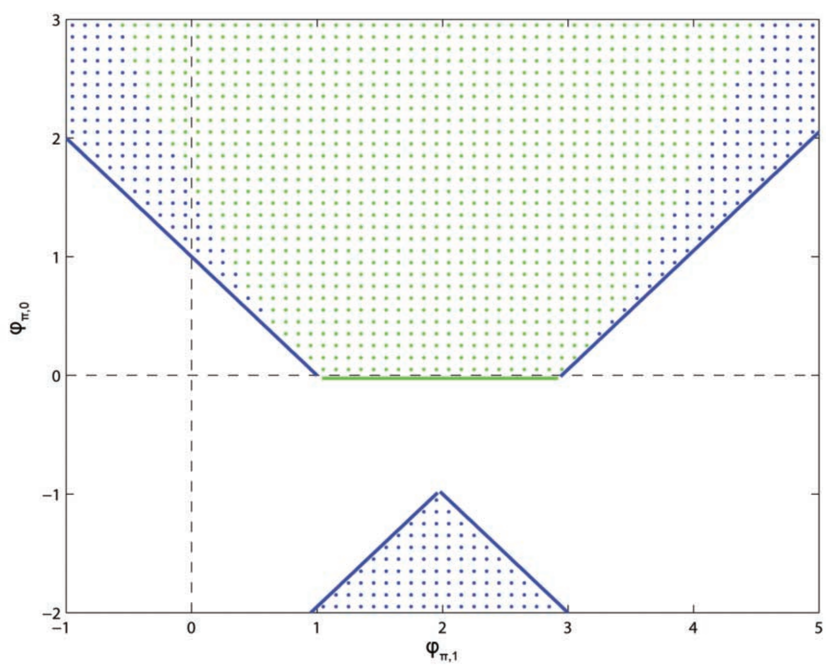

FIG. B5.—Zones of local determinacy when the policy rule responds to both current and expected future inflation, $i_t = \phi_{\pi,0}\pi_t + \phi_{\pi,1}\pi_{t+1}$, using $\beta = 0.95$ and $\sigma\gamma = 2$. The plotted boundaries are $\phi_{\pi,0} + \phi_{\pi,1} = 1$, $\phi_{\pi,1} - \phi_{\pi,0} = 1 + 2(1+\beta)/\sigma\gamma$, and $\phi_{\pi,0} = -(1-\beta)/\sigma\gamma$.

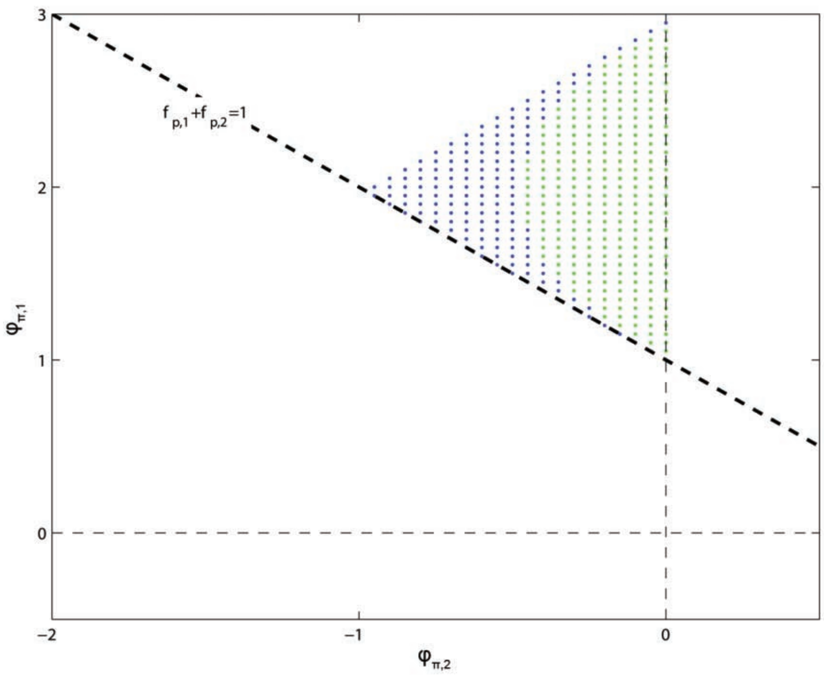

FIG. B6.—Region of local determinacy when the policy rule responds to expected future inflation one and two periods ahead, $i_t = \phi_1 E_t \pi_{t+1} + \phi_2 E_t \pi_{t+2}$, using $\beta = 0.95$, $\sigma\gamma = 2$.

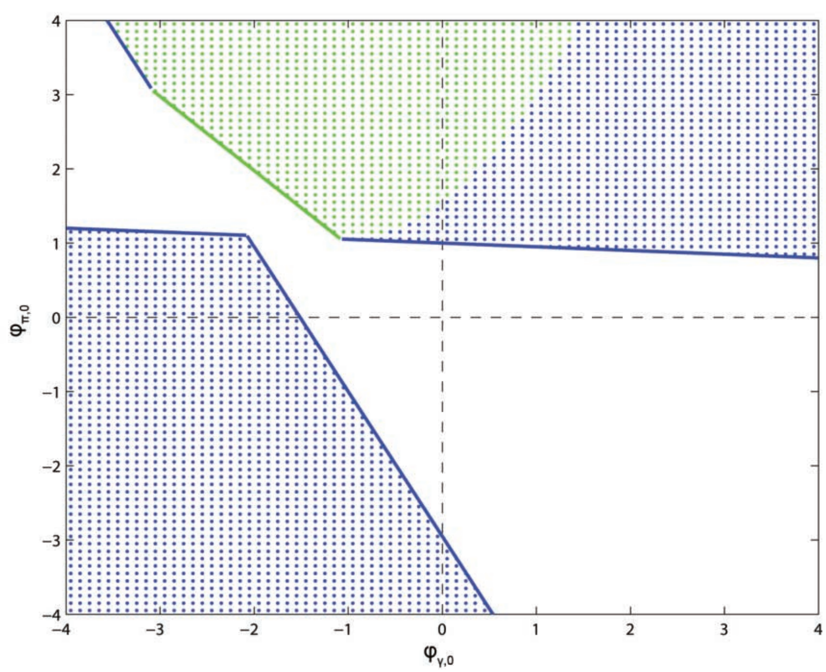

FIG. B7.—Region of determinacy in the three-equation model for a rule $i_t = \phi_{\pi,0}\pi_t + \phi_{y,0} y_t$, using $\beta = 0.95$, $\sigma = 2$, $\gamma = 1$.

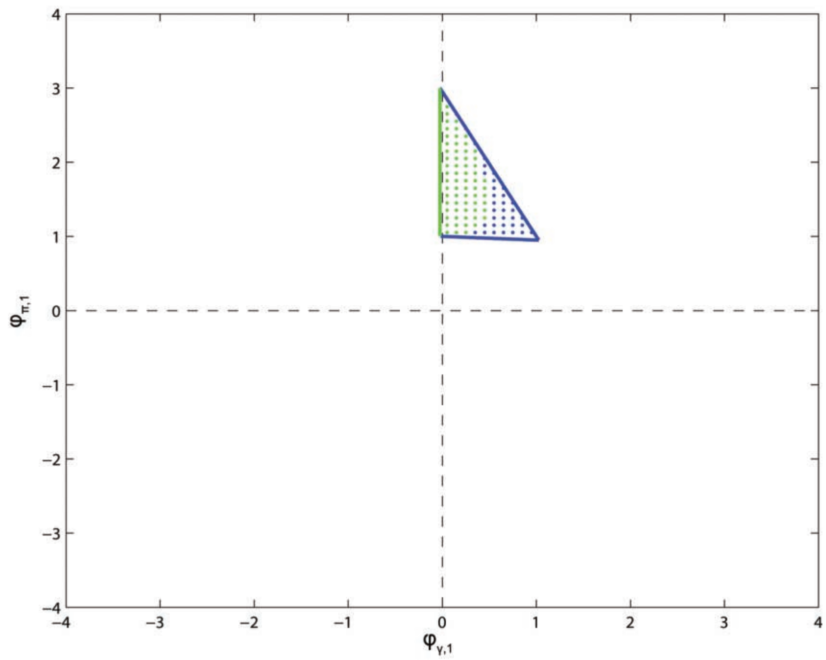

FIG. B8.—Region of determinacy for the three-equation model, for an interest rate rule $i_t = \phi_{\pi,1}E_t\pi_{t+1} + \phi_{y,1}E_ty_{t+1}$, using $\beta = 0.95$, $\sigma = 2$, $\gamma = 1$.

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