Answers:

1. VAR

   (a)

$$\begin{bmatrix} r_{t+1} \\ \Delta d_{t+1} \\ dp_{t+1} \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \\ 0.94 \end{bmatrix} dp_t + \begin{bmatrix} \varepsilon^r_{t+1} \\ \varepsilon^d_{t+1} \\ \varepsilon^{dp}_{t+1} \end{bmatrix}$$

$$\begin{bmatrix} \sigma(\varepsilon^r) & \rho_{r,d} & \rho_{r,dp} \\ - & \sigma(\varepsilon^d) & \rho_{d,dp} \\ - & - & \sigma(\varepsilon^{dp}) \end{bmatrix} = \begin{bmatrix} 0.20 & 0.7 & -0.7 \\ - & 0.14 & 0 \\ - & - & 0.15 \end{bmatrix}$$

Clearly, $r_{t+1} = -\rho dp_{t+1} + dp_t + \Delta d_{t+1}$ is going to help! If you forgot one of $b_r, b_d, \phi$, you can find the other from $b_r = -\rho\phi + b_d$.

I hope you remember $\sigma(r) \approx 0.20$, any number between 0.10 and 0.25 is fine, and $\rho_{\varepsilon^d,\varepsilon^{dp}} = 0$. Then $\varepsilon^r = -\rho\varepsilon^{dp} + \varepsilon^d$ gets you the rest. This could get you $\sigma^2(\varepsilon^r)$ if you forgot, $\sigma^2(\varepsilon^r) = \rho^2\sigma^2(\varepsilon^{dp}) + \sigma^2(\varepsilon^d)$

$$\sigma(\varepsilon^r) = \sqrt{0.96^2 \times 0.15^2 + 0.14^2} = 0.20$$

since you don't have a calculator,

$$\sigma(\varepsilon^r) \approx \sqrt{0.92 \times 0.0225 + 0.0196} \approx \sqrt{0.0200 + 0.0200} \approx \sqrt{0.0400} = 0.20$$

Now the correlations,

$$\varepsilon^r = -\rho\varepsilon^{dp} + \varepsilon^d$$

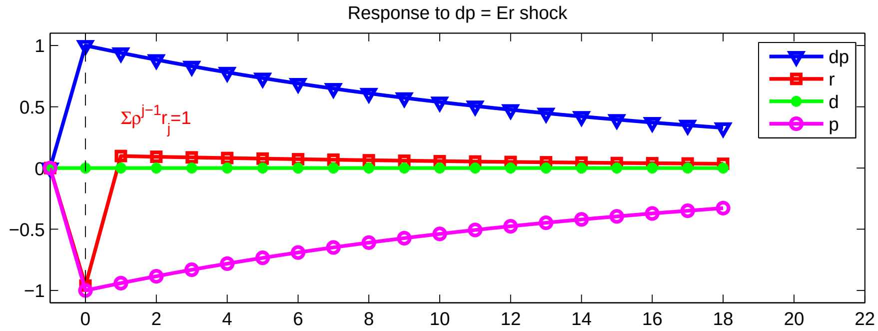
$$\begin{aligned} \frac{cov\left(\varepsilon^r, \varepsilon^{dp}\right)}{\sigma(\varepsilon^r)\sigma(\varepsilon^{dp})} &= \frac{-\rho\sigma^2\left(\varepsilon^{dp}\right) + \sigma\left(\varepsilon^d, \varepsilon^{dp}\right)}{\sigma(\varepsilon^r)\sigma(\varepsilon^{dp})} \\ &= \frac{-0.96 \times 0.15^2}{0.15 \times 0.20} = \frac{-0.96 \times 0.15}{0.20} = -0.72 \\ ( &\approx \frac{-0.14}{0.20} = -0.7) \end{aligned}$$

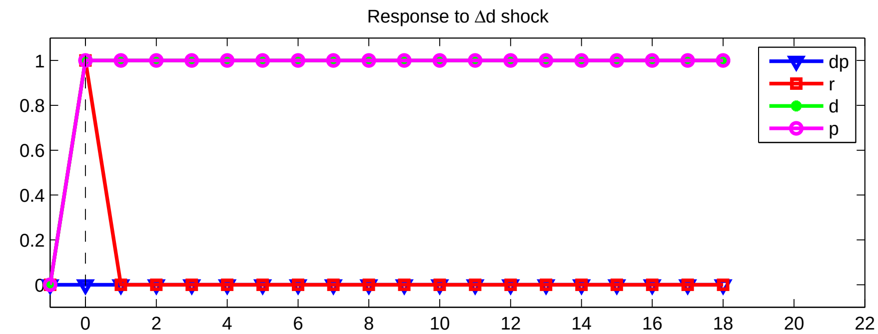
$$\begin{aligned} \frac{cov\left(\varepsilon^r, \varepsilon^d\right)}{\sigma(\varepsilon^r)\sigma(\varepsilon^d)} &= \frac{-\rho\sigma^2\left(\varepsilon^{dp}, \varepsilon^d\right) + \sigma^2(\varepsilon^d)}{\sigma(\varepsilon^r)\sigma(\varepsilon^d)} \\ &= \frac{0.14^2}{0.20 \times 0.14} = \frac{0.14}{0.20} = 0.7 \end{aligned}$$

   (b) i. "Cash flow" shock: $\begin{bmatrix} \varepsilon^r_{t+1} & \varepsilon^d_{t+1} & \varepsilon^{dp}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$

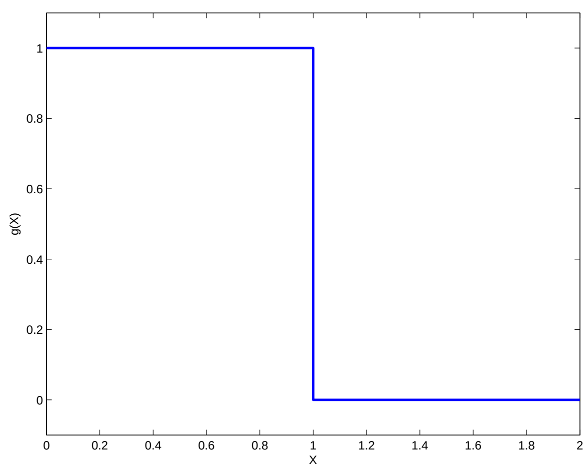
   ii. "Discount rate" shock: $\begin{bmatrix} \varepsilon^r_{t+1} & \varepsilon^d_{t+1} & \varepsilon^{dp}_{t+1} \end{bmatrix} = \begin{bmatrix} -0.96 & 0 & 1 \end{bmatrix}$

   (c) Different signs are fine, you are not expected to put the p shock in




2. It's a step function.




If you remembered the problem set, $g(X) = f''(X)$ up to boundary conditions which fortunately don't matter here. You'd get the answer. To check,

$$
\begin{aligned}
f(S_T) &= \int_0^{S_T} I_{0,b}(S_T - X)dX \\
&= \int_0^{b} (S_T - X)dX \\
S_T &< b : \int_0^{S_T} (S_T - X)dX = S_T^2 - \frac{1}{2}S_T^2 = \frac{1}{2}S_T^2
\end{aligned}
$$

$$b \quad < \quad S_T : \int_0^b (S_T - X) dX = S_T b - \frac{1}{2} b^2$$

3. Bonds

   (a)

$$\begin{aligned} P_t^{(1)} &= E_t M_{t+1} = e^{-x_t} \\ y_t^{(1)} &= x_t \end{aligned}$$

$$\begin{aligned} P_t^{(2)} &= E_t \left( M_{t+1} P_{t+1}^{(1)} \right) = E_t e^{-x_t - \frac{1}{2}\lambda^2 x_t^2 \sigma^2 - \lambda x_t v_{t+1}} e^{-x_{t+1}} \\ &= E_t e^{-x_t - \frac{1}{2}\lambda^2 x_t^2 \sigma^2 - \lambda x_t v_{t+1}} e^{-\phi x_t - v_{t+1}} \\ &= E_t e^{-(1+\phi) x_t - \frac{1}{2}\lambda^2 x_t^2 \sigma^2 - (1+\lambda x_t) v_{t+1}} \\ &= e^{-(1+\phi) x_t - \frac{1}{2}\lambda^2 x_t^2 \sigma^2 + \frac{1}{2}(1+\lambda x_t)^2 \sigma^2} \\ p_t^{(2)} &= -(1+\phi) x_t + \frac{1}{2} \left(1 + 2\lambda x_t\right) \sigma^2 \\ p_t^{(2)} &= \frac{1}{2}\sigma^2 - (1 + \phi - \lambda \sigma^2) x_t \end{aligned}$$

$$\begin{aligned} f_t^{(2)} &= p_t^{(1)} - p_t^{(2)} \\ f_t^{(2)} &= -\frac{1}{2}\sigma^2 + (\phi - \lambda\sigma^2) x_t \end{aligned}$$

   so

   (b)

$$\phi^* = (\phi - \lambda\sigma^2)$$

$$f_t^{(3)} = () + (\phi - \lambda\sigma^2)^2 x_t$$

   (c) Limits on $\lambda$ :

      i. $\lambda < 0$

      ii. If $\phi^* > \phi$, forward rates that converge faster than $\phi$, or spot rates that are "too sluggish" relative to $\lambda < 0$. How much less can we go?

$$\begin{aligned} \phi - \lambda\sigma^2 &< 1 \\ -(1-\phi) &< \lambda\sigma^2 \end{aligned}$$

      $\phi^* = \phi; \lambda = 0$ of course means the expexctations pattern, $f^{(n)} = \phi^{n-1}$

      iii. $\phi > \phi^* > 0$. The limit $\phi^* > 0$ implies

$$\lambda\sigma^2 < \phi$$

   (d)

$$f_t^{(2)} - y_t^{(1)} = -\frac{1}{2}\sigma^2 - \left[1 - \left(\phi - \lambda\sigma^2\right)\right] x_t$$

$$
\begin{aligned}
r_{t+1}^{(2)} - y_t^{(1)} &= p_{t+1}^{(1)} - p_t^{(2)} + p_t^{(1)} \\
r_{t+1}^{(2)} - y_t^{(1)} &= -\left(y_{t+1}^{(1)} - y_t^{(1)}\right) + \left(f_t^{(2)} - y_t^{(1)}\right) \\
r_{t+1}^{(2)} - y_t^{(1)} &= -\left(x_{t+1} - x_t\right) - \frac{1}{2}\sigma^2 - \left[1 - \left(\phi - \lambda\sigma^2\right)\right] x_t \\
r_{t+1}^{(2)} - y_t^{(1)} &= (1-\phi)x_t - v_{t+1} - \frac{1}{2}\sigma^2 - \left[1 - \left(\phi - \lambda\sigma^2\right)\right] x_t \\
r_{t+1}^{(2)} - y_t^{(1)} &= -\frac{1}{2}\sigma^2 - \lambda\sigma^2 x_t - v_{t+1}
\end{aligned}
$$

In terms of forward rates,

$$
\begin{aligned}
r_{t+1}^{(2)} - y_t^{(1)} &= -\frac{1}{2}\sigma^2 - \lambda\sigma^2 \frac{f_t^{(2)} - y_t^{(1)} + \frac{1}{2}\sigma^2}{-\left[1 - \phi + \lambda\sigma^2\right]} - v_{t+1} \\
r_{t+1}^{(2)} - y_t^{(1)} &= -\frac{1}{2}\left(\frac{1-\phi}{1-\phi+\lambda\sigma^2}\right)\sigma^2 + \frac{\lambda\sigma^2}{1-(\phi-\lambda\sigma^2)}\left(f_t^{(2)} - y_t^{(1)}\right) - v_{t+1}
\end{aligned}
$$

(e) Here, we need $\lambda > 0$ to produce even the sign of the FB coefficient, within the bound $\phi^* < 1$. At the bound $\phi^* = 0$ we have

$$
\frac{\lambda\sigma^2}{1-(\phi-\lambda\sigma^2)} = \phi < 1
$$

That's as large as we can get. We cannot produce $b = 1$ with sensible $\lambda$ in this model.

(f) Comments: In "Asset Pricing" I made graphs showing how the FB coefficients implied "sluggish adjustment," i.e. that forward rates converge *faster* than the yield autoregression suggests. In "Decomposing the yield curve," however, we see in the cross section that forward rates in fact imply "too fast" adjustment, that forward rates are flatter than yield regressions suggest. The answer? This is too simple a model to capture FB regressions!

Note, we can also look at the yield regressions

$$
\left(r_{t+1}^{(2)} - y_t^{(1)}\right) + \left(y_{t+1}^{(1)} - y_t^{(1)}\right) = \left(f_t^{(2)} - y_t^{(1)}\right)
$$

$$
\begin{aligned}
y_{t+1}^{(1)} - y_t^{(1)} &= (\phi - 1)x_t + v_{t+1} \\
y_{t+1}^{(1)} - y_t^{(1)} &= -(1-\phi)\frac{f_t^{(2)} - y_t^{(1)} + \frac{1}{2}\sigma^2}{-\left[1-\phi+\lambda\sigma^2\right]} + v_{t+1} \\
y_{t+1}^{(1)} - y_t^{(1)} &= +\frac{1}{2}\frac{(1-\phi)}{1-\phi+\lambda\sigma^2}\sigma^2 + \frac{1-\phi}{1-(\phi-\lambda\sigma^2)}\left(f_t^{(2)} - y_t^{(1)}\right) + v_{t+1}
\end{aligned}
$$

With $\lambda = 0$, this coefficient is 1, so the expectations hypothesis holds. With $\lambda < 0$, the coefficient is larger than one; with $\lambda > 0$, the coefficient is less than one

4. Hansen-Richard

(a) The set of returns is $R_t^f + w_t R_{t+1}^e$ and the set of excess returns is $w_t R_{t+1}^e$. There are only two assets, but there is a lot of conditioning information. Key confusions: there is *one* payoff space, not a different one for each information set, and agents can form portfolios by making conditional choices. Managed portfolios expand the set of payoffs.

(b) Choose any defining property of $R^*$ and $R^{e*}$ and work out what the weights $w_t$ must be. For $R^{e*} = w_t R^e$,you could remember or rederive the formula

$$\begin{aligned} R^{e*} &= E_t(R^e)' E_t(R^e R^{e\prime}) R^e \\ R^{e*} &= \frac{\mu}{\mu^2+\sigma^2} R^e \end{aligned}$$

Or, start from scratch using the defining property

$$\begin{aligned} E_t(R^{e*}R^e) &= E_t(R^e) \\ E_t(w_t R^{e2}) &= E_t(R^e) \\ w_t &= \frac{\mu}{\mu^2+\sigma^2} \end{aligned}$$

For $R^*$, if you remember

$$R_t^f = R^* + R_t^f R^{e*}$$

then

$$R^* = R_t^f - R_t^f R^{e*} = R_t^f - \frac{R_t^f \mu}{\mu^2+\sigma^2} R^e$$

Or, you can find it directly from any defining property. You know for any $R^e$,

$$\begin{aligned} 0 &= E_t(R^* R^e) \\ 0 &= E_t(\left(R^f + w_t R^e\right) R^e) \\ 0 &= R_t^f \mu + w_t \left(\mu^2+\sigma^2\right) \\ w_t &= -\frac{R_t^f \mu}{(\mu^2+\sigma^2)} \\ R^* &= R_t^f - R_t^f \frac{\mu}{\mu^2+\sigma^2} R_{t+1}^e \end{aligned}$$

Starting with $E(R^{*2}) = E(R^* R^f)$ also works.

(c) The UCMVF is

$$R^{mv} = R^* + w R^{e*}$$

In terms of the basis assets

$$R^{mv} = R_t^f - \frac{R_t^f \mu}{\mu^2+\sigma^2} R^e + w \frac{\mu}{\mu^2+\sigma^2} R^e$$

$$R^{mv} = R_t^f + (w - R_t^f) \frac{\mu}{\mu^2+\sigma^2} R^e$$

for *fixed* $w$. (i.e. not $w_t$) To generate $R^{mv} = R^f$ in state 1, then we must have $w = 1$. So, in state2, we must be looking for

$$R_t^f + (1 - R_t^f) \frac{\mu}{\mu^2+\sigma^2} R^e$$

$$R_t^f + (1 - R_t^f) \frac{\mu/\sigma}{\mu^2/\sigma^2 + 1} \frac{1}{\sigma} R^e$$

$$R_t^f + (-0.2)\frac{1}{2}\frac{1}{0.2}R^e$$
$$R_t^f - \frac{1}{2}R^e$$

$$R^{mv} = 1.20 - \frac{1}{2}R^e$$

so it's just halfway down. By choosing less mean you get less unconditional variance by getting less variance of conditional mean than you would have with Rf in both states