Fiscal theory with negative interest rates

John H. Cochrane[*]

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In this note, I show how fiscal theory can survive even in economies with $r < g$. Normal representative agent economies don't allow $r < g$. Bonds or money in the utility function can drive the interest rate below zero, or allowing growth, below the economic growth rate $r < g$. Nonetheless fiscal theory can determine the price level.

A particularly vivid case is a government financed only by money, which pays no interest. Here, $r = -\pi$ which is often negative and less than the growth rate. Nonetheless, fiscal theory can determine the price level, even though money is never "repaid" with surpluses.

This note is inspired by Kaplan, Nikolakoudis and Violante (2023). They generate lower interest rates via buffer stock savings of heterogenous agents, and via bonds in the utility function in an appendix. This note is a simplified version of the latter treatment, with my interpretation of the results.

The basic idea

The issue: The real value of debt is the present value of primary surpluses. In a steady state the real value of debt b, primary surplus s and interest rate r satisfy

$$b = \frac{s}{r}.$$

With $b > 0$, there are steady states with $r < 0$ and $s < 0$. Negative interest costs can finance a perpetual small primary deficit. But this is weird – a positive value of a negative cashflow divided by a negative interest rate?

[*]Hoover Institution, Stanford University and NBER. john.cochrane@stanford.edu, https://www.johnhcochrane.com.

It is weird, because of dynamics. The government budget satisfies

$$\frac{db_t}{dt} = r_t b_t - s_t.$$

Debt grows at the real rate, and declines with primary surpluses. Around a conventional $s > 0$ $r > 0$ steady state, debt generically explodes. Debt $b = s/r$ equal to the present value of surpluses is the only non-explosive solution, the only solution that satisfies the transversality condition. The price level at time 0 jumps so that $b_0 = B_0/P_0 = b$, and fiscal theory of the price level works.

However, if s and r are both negative, then

$$\frac{db_t}{dt} = rb_t - s$$

converges to $b = s/r$ for any b_0. We lost the forward looking, explosive eigenvalue of the system. The transversality condition is always satisfied. The present value of debt no longer determines the initial price level. Fiscal theory falls apart.

But with a liquidity / insurance / money value of debt, the interest rate r_t is not a constant. When there is more debt (more money), the equilibrium cost to households of holding debt declines, and the interest rate rises closer to the discount rate, i.e. the interest rate of the frictionless economy. When there is less debt, people are willing to pay a large opportunity cost to hold it, and the interest rate falls. Now dynamics are

$$\frac{db_t}{dt} = r_t(b_t)b_t - s \tag{1}$$

with $dr(b)/db > 0$. In this case,

$$\frac{d[r(b)b]}{db} = r(b) + \frac{dr(b)}{db}b.$$

If $dr(b)/db$ is large enough, the right side of (1) rises with b_t. More debt raises the interest rate and leads to higher subsequent debt, even if the interest rate itself is negative. Then we restore explosive local dynamics and the transversality condition selecting equilibria. Debt is, uniquely, the present value of surpluses, even with negative interest rate and negative surpluses!

More carefully, with bonds in utility

Bonds or money in the utility function are a nice way to generate a liquidity value of government debt. (Uninsured idiosyncratic risk and an absence of other assets also works, though with more algebra.)

Households maximize

$$\max \int e^{-\rho t}\left[u(c_t)+\theta v\left(a_t\right)\right]dt$$

subject to

$$\frac{da_t}{dt}=r_t a_t-\tau_t-c_t.$$

For examples, I use power utility

$$\int e^{-\rho t}\left[\frac{c_t^{1-\gamma}}{1-\gamma}+\theta\frac{(a_t)^{1-\eta}}{1-\eta}\right]dt$$

and usually log utility $\gamma=\eta=1$.

The consumer first order conditions (algebra below) are

$$\gamma_t\frac{1}{c_t}\frac{dc_t}{dt}=r_t-\rho+\frac{\theta v'(a_t)}{c_t^{-\gamma}}$$

where

$$\gamma_t\equiv-\frac{u''(c_t)}{c_t u'(c_t)}$$

and the transversality condition

$$\lim_{T\to\infty}e^{-\rho T}u'(c_t)a_t=0.$$

The real value of government debt satisfies

$$\frac{db_t}{dt}=r_t b_t-s_t.$$

I study a perfect foresight economy after an initial time-0 shock. Nominal debt B_0 is outstanding before and just after the shock. The price level P_0 adjusts, so $b_0=B_0/P_0$ is determined by nominal debt B_0 and the price level jump.

Debt is actually nominal, and follows

$$\frac{dB_t}{dt}=i_t B_t-P_t s_t. \tag{2}$$

The central bank sets an interest rate target i_t. (2) describes how the central bank can set an interest rate target. The bank offers to borrow or lend at rate i_t, but does not affect surpluses s_t. (2) then just describes what the amount of nominal debt will be. This equation thus implements in continuous time the story in *Fiscal Theory of the Price Level*, about how offering to sell debt at a constant interest rate implements share splits.

The real interest rate is determined by real considerations, and then inflation follows by

$$\pi_t = r_t + i_t.$$

In this flex price economy with short-term debt and no uncertainty past the initial shock, the interest rate simply sets inflation (equal to expected inflation) and fiscal policy determines the time-0 price level jump.

The "bonds" can thus also represent money. Money is just a form of government debt that happens to pay zero interest, the same thing as a bond with a zero interest rate peg. In this model, the only effect of relabeling bonds vs. money is that money constrains the interest rate target to be zero. In the following, we just find the initial price level jump and the path of real interest rates. Inflation then follows the interest rate target but has no other effect.

In equilibrium, $a_t = b_t$; $\tau_t = s_t$. I specify constant consumption $c_t = 1$ and surplus $s_t = s$. Thus the units of debt b_t are the debt to consumption ratio, with a number on the order of $b = 1$; the units of surplus s are surplus to consumption ratio. Below I generalize to growth, $c_t = e^{gt}$; $s_t = s_0 e^{gt}$ and $r < g$ issues. For now, it's a little simpler to look at the non-growing economy and the $r < 0$ case.

Thus equilibria satisfy two conditions

$$r_t = \rho - \theta v'(b_t) \tag{3}$$

$$\frac{db_t}{dt} = r_t b_t - s. \tag{4}$$

In the first condition we see how lower debt levels raise the marginal utility of debt and thus lower the equilibrium interest rate, potentially below zero.

One can substitute out the interest rate, and analyze the system with a single state variable,

$$\frac{db_t}{dt} = \left[\rho - \theta v'(b_t)\right] b_t - s$$

However, it's insightful to keep track of the interest rate and solve the system in a phase diagram, as we often do for systems of coupled differential equations.

Steady states

The steady states of (3)-(4) are

$$r = \rho - \theta v'(b)$$
$$r = \frac{s}{b}.$$

Following Kaplan, Nikolakoudis and Violante (2023), I'll call the former the steady state "demand" for debt, and the latter the steady-state "supply." Demand is a rising function of debt b, with asymptote $r = \rho$. With $s > 0$, supply is a declining function of b. We can also write it as $b = s/r$, the value of debt is the present value of primary surpluses. The interest rate is negative if and only if the surplus is negative.

Stationary equilibria are where supply and demand intersect, the b that solve

$$\frac{s}{b} = \rho - \theta v'(b). \tag{5}$$

With log utility $v'(b) = 1/(b)$,

$$\frac{s}{b} = \rho - \frac{\theta}{b}. \tag{6}$$

The solution is

$$b = \frac{s + \theta}{\rho}; \; r = \frac{s\rho}{s + \theta}. \tag{7}$$

Since we must have positive debt, $b > 0$, there is a limit to how much negative surplus can be financed: We must have $s > -\theta$ for an equilibrium.

Figure 1 illustrates. In the top equilibrium, I intersect supply with a positive surplus $s = 0.02$ with demand absent a liquidity effect, $\theta = 0$. The interest rate equals the discount rate, and debt is the present value of surpluses at that rate, $b = s/r = 0.02/0.05 = 0.4$. The middle (vertically) equilibrium raises the liquidity value of debt to $\theta = 0.03$. The equilibrium features a lower interest rate, and consequently greater value of debt.

The lower equilibrium maintains the same liquidity value, $\theta = 0.03$, but contemplates a negative surplus, $s = -0.01$. Here too, there is a unique steady state, with debt $b = (-0.01 + 0.03)/0.05 = 0.4$. The interest rate is negative, $r = s/b = -0.01/0.4 = -0.025$ or -2.5%.

But does this equilibrium make sense? A present value interpretation of $b = s/r$ with two negative numbers on the right hand side looks weird. To understand that issue, turn to the dynamics.

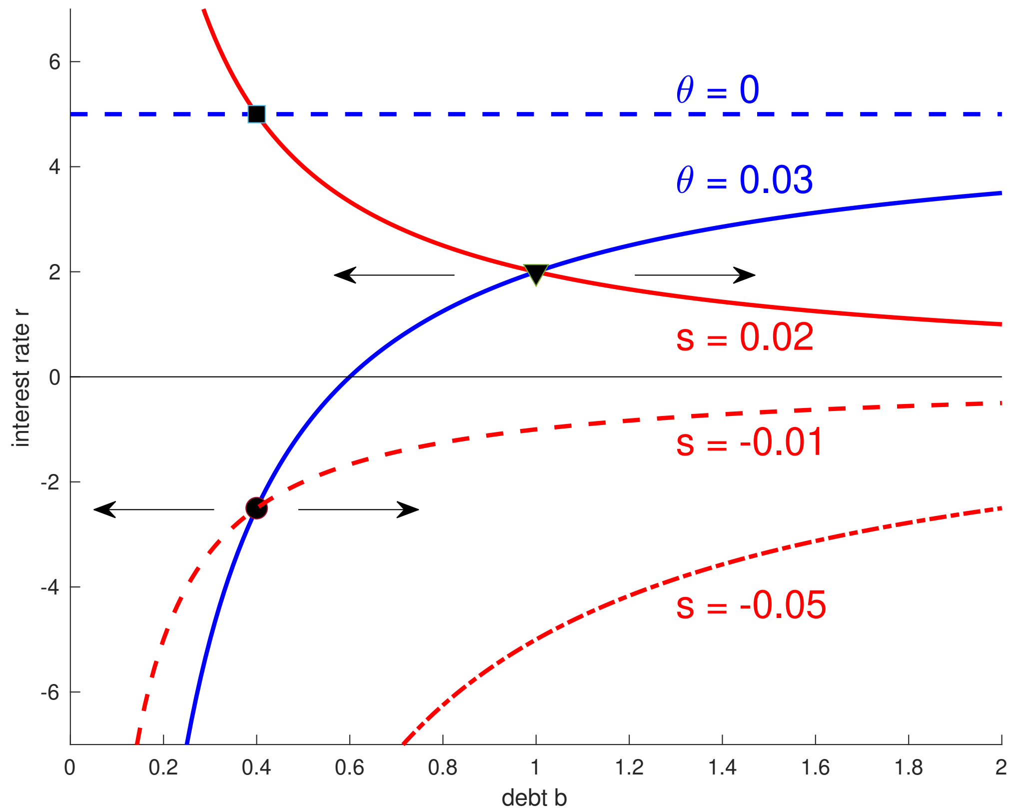

Figure 1: Debt and interest rates with debt in the utility function and log utility. Parameters $\rho = 0.05$, s and θ as indicated.

Dynamics

Repeating for convenience, the dynamic equilibrium conditions (3)-(4) are

$$r_t = \rho - \theta v'(b_t)$$
$$\frac{db_t}{dt} = r_t b_t - s$$

With no utility of debt, $\theta = 0$, $r = \rho > 0$, and

$$\frac{db_t}{dt} = r b_t - s.$$

Solutions generically explode. Debt $b = s/\rho = s/r$ is the present value of surpluses, is the only non-explosive solution. The price level at time 0 jumps so that $b_0 = B_0/P_0 = b$, and fiscal theory of the price level works.

However, if s and r are both negative and r is constant, then

$$\frac{db_t}{dt} = rb_t - s$$

converges to $b = s/r$ for any b_0. We lost the forward looking, explosive eigenvalue of the system. The transversality condition is always satisfied. The present value of debt no longer determines the initial price level. Fiscal theory falls apart. Explicitly, solutions are

$$b_t = \left(b_0 - \frac{s}{r}\right) e^{rt} + \frac{s}{r}$$

which with $r < 0$ converge for any b_0.

But the interest rate r_t is not a constant. With $\theta > 0$, we have

$$\frac{db_t}{dt} = \left[\rho - \theta v'(b_t)\right] b_t - s \tag{8}$$

Higher b_t means lower $v'(b_t)$ and thus higher db/dt. Higher b_t lowers the liquidity value of debt, increasing an off-steady-state explosion. And if $-\theta v'(b_t)$ rises fast enough with b_t, the local dynamics can be explosive even if $r = \rho - \theta v'(b) < 0$.

We can see these local dynamics with a first order approximation of (8) around the steady state,

$$\frac{db_t}{dt} \approx \left[r - \theta v''(b) b\right] (b_t - b) . \tag{9}$$

(Here I use $\rho - \theta v'(b) = r$.) Negative v'', declining marginal utility of bonds, raises the coefficient on $(b_t - b)$. If $v''(b)$ is a large enough negative number, the local dynamics are explosive, $r - \theta v''(b) b > 0$, even if $r < 0$. Then we have a unique solution and fiscal theory works, even with a *negative* interest rate.

In our log utility example, the dynamics (8) are

$$\frac{db_t}{dt} = \left[\rho - \theta b_t^{-1}\right] b_t - s$$

$$\frac{db_t}{dt} = \rho b_t - \theta - s \tag{10}$$

and the linear approximation (9) is exact,

$$\frac{db_t}{dt} = \rho \left(b_t - b\right)$$

The steady state is explosive for *any* θ, s. (Recall, though, that we must have $s > \theta$ for a steady state to exist.) The arrows in Figure 1 indicate the local instability of the equilibria. Therefore fiscal theory still works, and the initial price level jumps so that $b_0 = b$ even with $s < 0$ and $r < 0$.

The graph suggests that the steady state equilibrium is locally unstable and hence unique if demand cuts supply downward from left to right. This hunch is correct. The slope of the supply curve $r = s/b$ is

$$\frac{dr}{db} = -\frac{s}{b^2}.$$

The slope of the demand curve $r = \rho - \theta v'(b)$ is

$$\frac{dr}{db} = -\theta v''(b) = \frac{\theta}{b^2},$$

the latter in the log utility case. Supply cuts downward at the steady state if

$$-\frac{s}{b^2} < -\theta v''(b)$$

or with log utility

$$-\frac{s}{b^2} < \frac{\theta}{b^2}.$$

Equivalently,

$$\frac{s}{b^2} > \theta v''(b);\ r > \theta v''(b)b$$

or with log utility

$$s > -\theta.$$

The former is exactly the condition for locally explosive or unique equilibria in (9); the latter the condition for existence of the steady state.

Flows and present values

Equation (10) shows that, with log utility, debt explodes at rate ρ, not at rate r. We should sitll discount with the positive discount rate, not the negative interest rate. Integrating (10) forward,

$$b_0 = \int_{t=0}^{\infty} e^{-\rho t}(\theta + s)dt = \frac{\theta + s}{\rho}$$

which is our steady state. In this epression, we can think of debt as providing a stream of liquidity services, θ. The liquidity services can be so large that they fund a negative stream of deficits. We have a well defined present value and fiscal theory applies.

We can also try to discount at the interest rate,

$$b_0 = \int_{t=0}^{\infty} e^{-(\rho-\theta/b_t)t} s dt.$$

This gives the same answer when the interest rate is positive, but does not work when the interest rate is negative. Thinking about liquidity as a distortion to the cash flow rather than to the rate of return allows us to use present value thinking.

This is an instance of a general and underappreciated theoretical point: In many situations, you can discount with marginal utility $e^{-\rho t}u'(c_t)/u'(c_0)$ to produce a sensible present value formula, but alternative discount factors that account for one period returns such as e^{-r_t} do not produce a convergent present value. Discount factor tricks that work in finite-horizon models do not work for infinite horizon models.

More generally, consider (8) with a varying surplus as well,

$$\frac{db_t}{dt} = \left[\rho - \theta v'(b_t)\right] b_t - s_t.$$

One way to express the solution is

$$b_0 = \int_{t=0}^{\infty} e^{-\rho t} \left[\theta v'(b_t) b_t + s_t\right] dt$$

Again, we discount the surplus and the liquidity services of debt at rate ρ, resulting in a well defined present value.

Equation (7) expresses this idea as well for the log utility case:

$$b = \frac{s+\theta}{\rho}.$$

The value of the debt is the value of surpluses that repay debt, plus the value of liquidity services. We can fund a slightly negative surplus when debt has adequate liquidity services.

Multiple or no equilibria.

The conditions for steady state

$$r = \rho - \theta v'(b)$$
$$r = \frac{s}{b}.$$

may have no solution, or may have multiple equilibria.

In the log utility case $v'(b) = 1/b$,

$$r = \rho - \frac{\theta}{b}$$
$$r = \frac{s}{b}.$$

With $s > 0$ and $\theta > 0$ there is one and only one solution. Supply slopes down and demand slopes up, both ranging from $r = 0$ to $r = \infty$.

However, with $s < 0$ there is no positive $(b > 0)$ solution if $s < -\theta$. Intuitively, writing the steady state as

$$b = \frac{\theta + s}{\rho},$$

liquidity benefits θ must pay for negative surpluses $-s$ leaving something positive for bondholders. Figure 1 displays this case as well. The supply curve is shifted over to the right and down, and the demand curve never catches up. If there is a solution it is unique. Thus, for log utility and $s < 0$ there is 0 or 1 solution, and if there is a solution it is unstable and determines the price level.

Kaplan, Nikolakoudis and Violante (2023) find multiple equilibria. Since both supply and demand slope upward with $s < 0$, there can be more equilibria if the demand curve is more concave than the supply curve. Power utility does not generate an example, but other utility functions may.

Following Kaplan, Nikolakoudis and Violante (2023), multiple equilibria occur if the marginal utility of debt is bounded. Then the demand curve in Figure 1 intersects the vertical axis. This occurs if the utility function for debt is modified to $\theta v(a_{+}\underline{a})$. Then marginal utility of debt b becomes

$$\theta v'(b + \underline{a}) = \frac{\theta}{b + \underline{a}},$$

With the latter holding for $v(a) = \log(a)$, which I use in what follows. Even at $b = 0$, the vertical axis of Figure 1, marginal utility is finite.

The steady state supply and demand curves are

$$r = \rho - \frac{\theta}{b + \underline{a}}$$

$$r = \frac{s}{b}.$$

The condition for a steady state is then

$$\frac{s}{b} = \rho - \frac{\theta}{b + \underline{a}}$$

$$s(b + \underline{a}) = (b + \underline{a})\rho b - \theta b$$

x

$$0 = \rho b^2 - [(s + \theta) - \underline{a}\rho]b - \underline{a}s$$

There are now potentially two solutions,

$$b = \frac{[(s + \theta) - \rho\underline{a}] \pm \sqrt{[(s + \theta) - \rho\underline{a}]^2 + 4\rho s\underline{a}}}{2\rho}.$$

Figure 2 plots this example. The blue demand curve shifts to the left by $\underline{a}$, and now intersects the vertical axis. Even with zero debt, the marginal utility of debt is bounded. Now there are two equilibria.

As you might expect, the low-debt equilibrium is stable, and multiple paths of debt approach it. The fiscal theory now implies only a lower bound on the price level, with that bound approaching the high debt steady state. Any larger price level P_0 leads to lower quantities of debt, that approach the low debt steady state.

Kaplan, Nikolakoudis and Violante (2023) suggest a variety of policies to remove the lower steady state and restore a unique price level. For example, they show that a surplus policy

$$s_t = s^* + \phi(b_t - b^*)$$

restores determinacy. With this modification, the red supply curve of steady states becomes

$$r_t = \frac{s^* + \phi(b - b^*)}{b} = \phi + \frac{s^* - \phi b^*}{b}.$$

For sufficient ϕ, the red supply curve now slopes down and intersects the blue demand curve once.

The trouble with this and related solutions is that they turn off equilibrium behavior on its head. Here the government commits to *reducing* surpluses as debt gets larger, and *increasing*

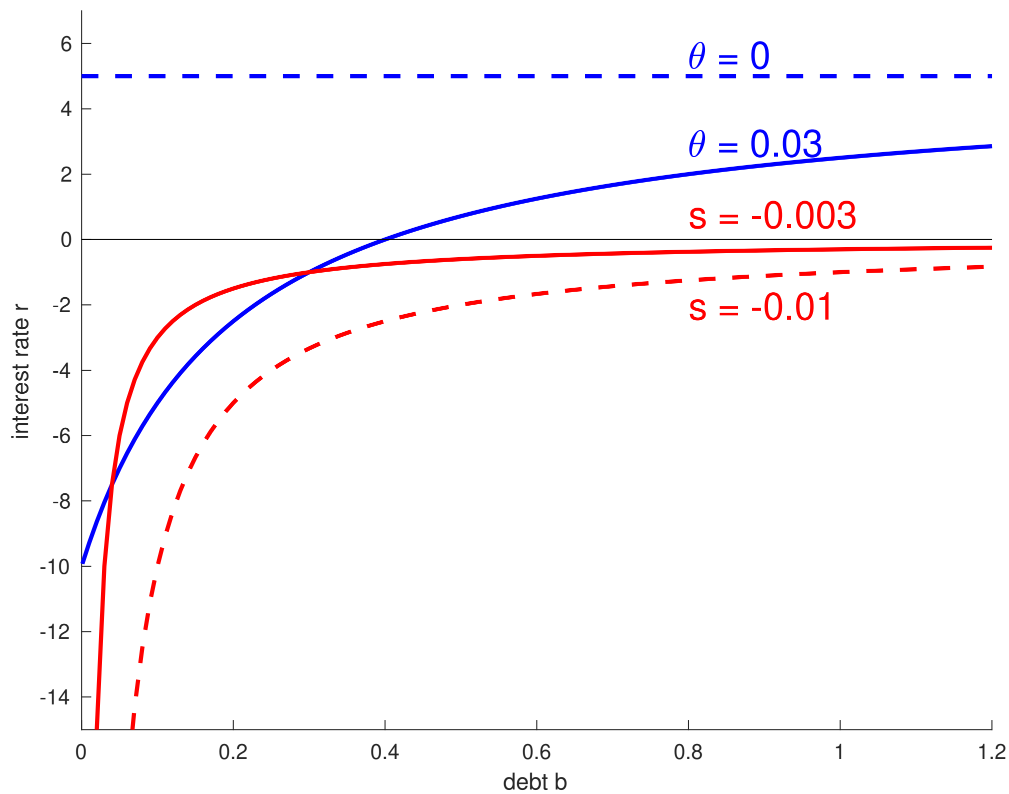

Figure 2: Debt and interest rates with debt in the utility function and log utility. Parameters $\rho = 0.05$, $\underline{a} = 0.20$, and s and θ as indicated.

surpluses as debt gets smaller, the opposite of usual policies to stabilize the price level. Well, the point is to induce instability. This has the flavor of new-Keynesian Taylor rules which likewise imagine the government deliberately destabilizes policy to attain determinacy.

For now, I leave this issue. Are these multiple equilibria important? They are not generic. They occur only when there is a liquidity value of government debt, a perpetual $r < g$, and for specific functional forms, with a limited liquidity value of debt.

This issue has a long tradition in monetary economics. What is money demand as the quantity of money goes to zero? One can make the argument for finite liquidity services, or unbounded. Or that the issue involves angels on heads of pins.

First order condition derivation

The first order conditions are

$$H = u(c_t) + \theta v(a_t) + \mu_t(r_t a_t - \tau_t - c_t)$$

$$\begin{aligned}\frac{\partial H}{\partial c} &: u'(c_t) = \mu_t \\ \frac{\partial H}{\partial a} &: \theta v'(a_t) + \mu_t r_t = -\frac{d\mu_t}{dt} + \rho \mu_t\end{aligned}$$

$$\begin{aligned}\frac{\partial H}{\partial c} &: u'(c_t) = \mu_t \\ \frac{\partial H}{\partial a} &: \theta v'(a_t) + r_t \mu_t = -u''(c_t)\frac{dc_t}{dt} + \rho u'(c_t)\end{aligned}$$

$$\frac{\theta v'(a_t)}{u'(c_t)} + r_t = -\frac{c_t u''(c_t)}{u'(c_t)}\frac{1}{c_t}\frac{dc_t}{dt} + \rho$$

$$\gamma \frac{1}{c_t}\frac{dc_t}{dt} = r_t - \rho + \frac{\theta v'(a_t)}{c_t^{-\gamma}}$$

and the transversality condition

$$\lim_{T \to \infty} e^{-\rho T} u'(c_t) a_t = 0.$$

Reference

Kaplan, Greg, Georgios Nikolakoudis and Giovanni L. Violante (2023) “Price Level and Inflation Dynamics in Heterogeneous Agent Economies,” Manuscript, University of Chicago.