# Portfolio Theory

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## Abstract

By simply reinterpreting the symbols, the familiar one period mean-variance portfolio theory can also apply to fully dynamic and intertemporal problems. This Chapter shows how. The centerpiece is a dynamic two-fund theorem. Intertemporal investors with quadratic period utility split their portfolios between a riskless asset and a risky asset. The riskless asset is an indexed consol, which pays a constant real coupon every period. The risky asset is a claim to the aggregate consumption stream. The risky asset, and all investors' optimal portfolios can be described as a "long run" version of a mean-variance frontier.

# 1 Introduction

Now we turn to one of the classic questions of finance–portfolio theory. Given a set of available assets, i.e. given their prices and the (subjective) distribution of their payoffs, what is the optimal portfolio? This is obviously an interesting problem, to Wall Street as well as to academics.

We can also view this problem as an alternative approach to the asset pricing question. So far, we have modeled the consumption process, and then found prices from marginal utility, following Lucas' (1978) "endowment economy" logic. We could instead model the *price process*, implicitly specifying linear technologies, and derive the optimal *quantities*, i.e. optimal portfolio holdings and the consumption stream they support. This is in fact the way asset pricing was originally developed.

I start by developing portfolio theory by the choice of *final payoff*. This is often a very easy way to approach the problem, and it ties portfolio theory directly into the $p = E(mx)$ framework of the rest of the book. Dynamic portfolio choice is, unsurprisingly, the same thing as static portfolio choice of managed portfolios, or contingent claims. I then develop the "standard approach" to portfolio theory, in which we choose the *weights* in a given set of assets, and I compare the two approaches.

# 2 One period portfolio problems, choosing payoffs

## 2.1 Complete markets

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The investor invests, and then values consumption at a later period. We summarize prices and payoffs by a discount factor $m$. We solve first order conditions $u'(c) = \lambda m$ for the optimal portfolio $c = u'^{-1}(\lambda m)$. If consumption is driven by an asset payoff $\hat{x}$ and outside income $e$, then $\hat{x} = u'^{-1}(\lambda m) - e$. The investor sells off outside income, then invests in a portfolio driven by contingent claims prices.

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Complete markets are the simplest case, and they can make the portfolio problem almost trivial. As usual, denote prices $p$, payoffs $x$. Given the absence of arbitrage opportunities there is a unique, positive stochastic discount factor or contingent claims price $m$ such that $p = E(mx)$ for any payoff $x$. This is a key step: rather than face the investors with prices and payoffs, we summarize the information in prices and payoff by a discount factor. That summary makes the portfolio problem much easier.

Now, consider an investor with utility function over terminal consumption $E[u(c)]$, initial wealth $W$ to invest, and random labor or business income $e$. The last ingredient is not common in portfolio problems, but I'll argue it's really important, and it's easy to put it in. The business or labor income $e$ is not directly tradeable, though there may be traded assets with similar payoffs that can be used to hedge. In a complete market, of course, there are assets that can perfectly replicate the payoff $e$.

The investor's problem is to choose a portfolio. Let's call the payoff of his portfolio $\hat{x}$, so its price or value is $p(\hat{x}) = E(m\hat{x})$. He will eat $c = \hat{x} + e$. Thus, his problem is

$$\max_{\{\hat{x}\}} E\left[u(\hat{x}+e)\right] \;\; s.t. \;\; E(m\hat{x}) = W \tag{1}$$

$\text{Max}_{\{\hat{x}\}}$ means "choose the payoff in every state of nature. In a discrete state space, this means

$$\max_{\{\hat{x}_i\}} \sum_i \pi_i u\left(\hat{x}_i + e\right) \;\; \text{s.t.} \;\; \sum_i \pi_i m_i \hat{x}_i = W$$

This is an easy problem to solve. The first order conditions are

$$u'(c) = \lambda m \tag{2}$$
$$u'\left(\hat{x}+e\right) = \lambda m. \tag{3}$$

*The optimal portfolio sets marginal utility proportional to the discount factor.* The optimal portfolio itself is then

$$\hat{x} = u'^{-1}(\lambda m) - e. \tag{4}$$

We find the Lagrange multiplier $\lambda$ by satisfying the initial wealth constraint. Actually doing this is not very interesting at this stage, as we are more interested in how the optimal portfolio distributes across states of nature than we are in the overall level of the optimal portfolio.

Condition (4) is an old friend. The discount factor represents contingent claims prices, so condition (2) says that marginal rates of substitution should be proportional to contingent claim price ratios. The investor will consume less in high price states of nature, and consume more in low price states of nature. Risk aversion, or curvature of the utility function, determines how much the investor is willing to substitute consumption across states. Equation (4) says that the optimal *asset* portfolio $\hat{x}$ first sells off, hedges or otherwise accommodates labor income $e$ one for one and then makes up the difference.

Condition (2) is the same first order condition we have been exploiting all along. If the investor values first period consumption $c_0$ as well and discounts future utility by $\beta$, then we know the marginal utility of first period consumption equals the shadow value of wealth, $\lambda = u'(c_0)$, so (2) becomes our old friend

$$\beta \frac{u'(c)}{u'(c_0)} = m.$$

We didn't really need a new derivation. We are merely taking the same first order condition, and rather than fix *consumption* and solve for *prices* (and returns, etc.), we are fixing *prices* and payoffs, and solving for *consumption* and the portfolio that supports that consumption.

### 2.1.1 Power utility and the demand for options

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For power utility $u'(c) = c^{-\gamma}$and no outside income, the return on the optimal portfolio is $\hat{R} = m^{-\frac{1}{\gamma}}/E(m^{1-\frac{1}{\gamma}})$ Using a lognormal iid stock return, this result specializes to $\hat{R} = e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)} R_T^{\alpha}$ where is the stock return and $\alpha \equiv \frac{1}{\gamma}\frac{\mu-r}{\sigma^2}$. The investor wants a payoff which is a nonlinear, power function of the stock return, giving rise to demands for options.

The same method quickly extends to a utility function with a "habit" or "subsistence level", $u'(c) = (c-h)^{-\gamma}$. This example gives a strong demand for put options.

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Let's try this idea out on our workhorse example, power utility. Ignoring labor income, the first order condition, equation (2), is

$$\hat{x}^{-\gamma} = \lambda m$$

so the optimal portfolio (4) is

$$\hat{x} = \lambda^{-\frac{1}{\gamma}} m^{-\frac{1}{\gamma}}$$

Using the budget constraint $W = E(m\hat{x})$ to find the multiplier,

$$\begin{aligned} W &= E(m\lambda^{-\frac{1}{\gamma}} m^{-\frac{1}{\gamma}}) \\ \lambda^{-\frac{1}{\gamma}} &= \frac{W}{E\left(m^{1-\frac{1}{\gamma}}\right)}, \end{aligned}$$

the optimal portfolio is

$$\hat{x} = W \frac{m^{-\frac{1}{\gamma}}}{E(m^{1-\frac{1}{\gamma}})}. \quad (5)$$

The $m^{-\frac{1}{\gamma}}$ term is the important one – it tells us how the portfolio $\hat{x}$ varies across states of nature. The rest just makes sure the scale is right, given this investor's initial wealth $W$.

In this problem, payoffs scale with wealth. This is a special property of the power utility function – richer people just buy more of the same thing. Therefore, the *return* on the optimal portfolio

$$\hat{R} = \frac{\hat{x}}{W} = \frac{m^{-\frac{1}{\gamma}}}{E(m^{1-\frac{1}{\gamma}})} \quad (6)$$

is independent of initial wealth. We often summarize portfolio problems in this way by the *return* on the optimal portfolio.

To apply this formula, we have to specify an interesting set of payoffs and their prices, and hence an interesting discount factor. Let's consider the classic Black-Scholes environment: there is a risk free bond and a single lognormally distributed stock. By allowing dynamic trading or a complete set of options, the market is "complete," at least enough for this exercise. (The next section discusses just how "complete" the market has to be.)

The stock, bond, and discount factor follow

$$\frac{dS}{S} = \mu dt + \sigma dz \quad (7)$$

$$\frac{dB}{B} = r dt \quad (8)$$

$$\frac{d\Lambda}{\Lambda} = -r dt - \frac{\mu - r}{\sigma} dz \quad (9)$$

(These are also equations (17.2) from Chapter 17, which discusses the environment in more detail. You can check quickly that this is a valid discount factor, i.e. $E(d\Lambda/\Lambda) = -r dt$ and $E(dS/S) - r dt = -E(d\Lambda/\Lambda \; dS/S)$). The discrete-time discount factor for time $T$ payoffs is $m_T = \Lambda_T/\Lambda_0$. Solving these equations forward and with a bit of algebra below, we can evaluate Equation (6),

$$\hat{R} = e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)} \; R_T^{\alpha}$$

where $R_T = S_T/S_0$ denotes the stock return, and

$$\alpha \equiv \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}.$$

($\alpha$ will turn out to be the fraction of wealth invested in stocks, if the portfolio is implemented by dynamic stock and bond trading.)

The optimal payoff is power function of the stock return. Figure 1 plots this function using standard values $\mu - r = 8\%$ and $\sigma = 16\%$ for a few values of risk aversion $\gamma$. For $\gamma = \frac{0.09-0.01}{0.16^2} = 3.125$, the function is linear – the investor just puts all his wealth in the stock. At lower levels of risk aversion, the investor exploits the strong risk-return tradeoff, taking a position that is much more sensitive to the stock return at $R_T = 1$. He gains enormous wealth if stocks go up (vertical distance past $R_T = 1$), and the cost of somewhat less consumption if stocks go down. At higher levels of risk aversion, the investor accepts drastically lower payoffs in the good states (on the right) in order to get a somewhat better payoff in the more expensive (high $m$) bad states on the left.

The optimal payoffs in Figure 1 are nonlinear. The investor does not just passively hold a stock and bond portfolio. Instead, he buys a complex set of contingent claims, trades dynamically, or buys a set of options, in order to create the nonlinear payoffs shown in the Figure. Fundamentally, this behavior derives from the nonlinearity of marginal utility, combined with the nonlinearity of the state-prices implied by the discount factor.

*Algebra.* The solutions of the pair (7)-(**??**) are (see (17.5) for more detail),

$$\ln S_T = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}\varepsilon \quad (10)$$

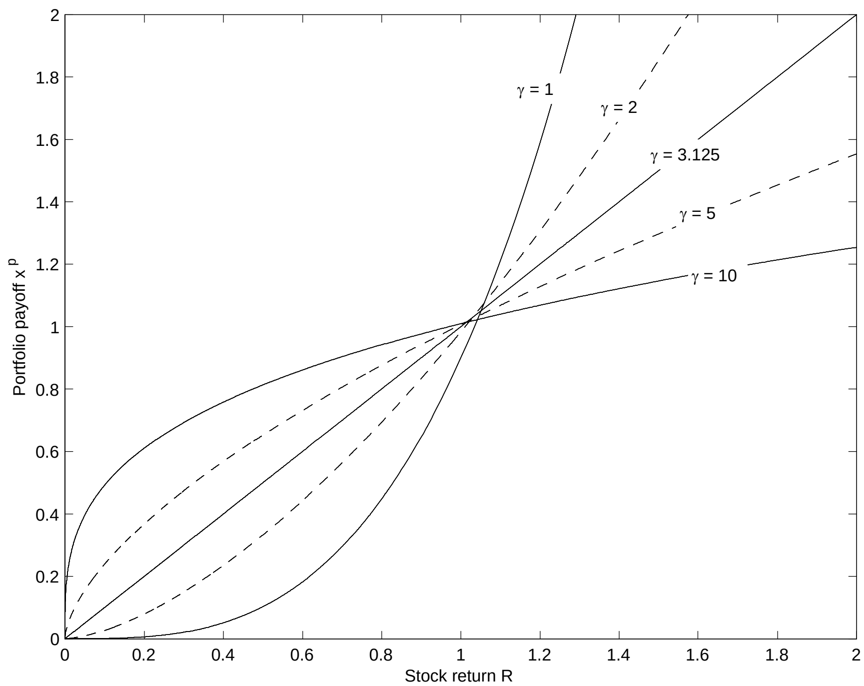


Figure 1: Return of an optimal portfolio. The investor has power utility $u\prime(c) = c^{-\gamma}$. He chooses an optimal portfolio in a complete market generated by a lognormal stock return with 9% mean and 16% standard deviation, and a 1% risk free rate.

$$\ln \Lambda_T = \ln \Lambda_0 - \left[ r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \right] T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon \tag{11}$$

with $\varepsilon \tilde{} N(0,1)$. We thus have

$$\begin{aligned} E\left(m_T^{1-\frac{1}{\gamma}}\right) &= \exp\left[ -\left(1-\frac{1}{\gamma}\right)\left[ r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \right] T + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \left(\frac{\mu - r}{\sigma}\right)^2 T \right] \\ &= \exp\left\{ -\left(1-\frac{1}{\gamma}\right)\left[ r + \frac{1}{2}\left(\frac{1}{\gamma}\right)\left(\frac{\mu - r}{\sigma}\right)^2 \right] \right\} T. \end{aligned}$$

Using $R_T = S_T/S_0$ to substitute out $\varepsilon$,

$$\begin{aligned} m_T^{-\frac{1}{\gamma}} &= \exp\left\{ \frac{1}{\gamma}\left[ r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \right] T + \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}\left[ \ln R_T - \left(\mu - \frac{\sigma^2}{2}\right) T \right] \right\} \\ &= \exp\left\{ \frac{1}{\gamma}\left[ r - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 - \frac{\mu - r}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right) \right] T + \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}\ln R_T \right\} \end{aligned}$$

Thus,

$$
\begin{aligned}
\hat{R} &= \exp\left[r - \frac{1}{2}\left(\frac{1}{\gamma^2}\right)\left(\frac{\mu-r}{\sigma}\right)^2 - \frac{1}{\gamma}\frac{\mu-r}{\sigma^2}\left(r-\frac{\sigma^2}{2}\right)\right]T \times \exp\left\{\frac{1}{\gamma}\frac{\mu-r}{\sigma^2}\ln R_T\right\} \\
&= \exp\left[r - \frac{1}{2}\sigma^2\alpha^2 - \alpha\left(r-\frac{\sigma^2}{2}\right)\right]T \times \exp\left\{\alpha \ln R_T\right\} \\
&= \exp\left[(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T\right] \times R_T^{\alpha}
\end{aligned}
$$

*Implementation*

This example will still feel empty to someone who knows standard portfolio theory, in which the maximization is stated over portfolio shares of specific assets rather than over the final payoff. Sure, we have characterized the optimal *payoffs*, but weren't we supposed to be finding optimal *portfolios*? What stocks, bonds or options does this investor actually hold?

Figure 1 *does* give portfolios. We are in a complete market. Figure 1 gives the number of contingent claims to each state, indexed by the stock return, that the investor should buy. In a sense, we have made the portfolio problem very easy by very cleverly choosing a simple basis – contingent claims – for our complete market.

There is a remaining largely technical question: suppose you wanted to implement this pattern of contingent claims by explicitly buying standard put and call options, or by dynamic trading in a stock or bond, rather than by buying contingent claims. How would you do it? I'll return to these questions below, and you'll see that they involve a more algebra. But really, they are technical questions. We've solved the important *economic* question, what the optimal payoff should *be*. Ideally, in fact, an intermediary (investment bank) would handle the financial engineering of generating most cheaply the payoff shown in Figure 1, and simply sell the optimal payoff directly as a retail product.

That said, there are two obvious ways to approximate payoffs like those Figure 1. First, we can approximate nonlinear functions by a series of linear functions. The low risk aversion ($\gamma = 1, \gamma = 2$) payoffs can be replicated by buying a series of call options, or by holding the stock and writing puts. The high risk aversion ($\gamma = 5, \gamma = 10$) payoffs can be replicated by writing call options, or by holding the stock and buying put options. The put options provide "portfolio insurance." Thus we see the demand and supply for *options* emerge from different attitudes towards risk. In fact many investors *do* explicitly buy put options to protect against "downside risk," while many hedge funds do, explicitly or implicitly, write put options.

Second, one can trade dynamically. In fact, as I will show below, the standard approach to this portfolio problem does not mention options at all, so one may wonder how I got options in here. But the standard approach leads to portfolios that are continually rebalanced. As it turns out, this payoff can be achieved by continually rebalancing a portfolio with $\alpha$ fraction of wealth held in stock. If you hold, say $\alpha = 60\%$ stocks and 40% bonds, then as the market goes up you will sell some stocks. This action leaves you less exposed to further stock market increases than you would otherwise be, and leads to the concave ($\gamma > 3.125$) discrete-period behavior shown in the graph.

*Example 2: Habits*

A second example is useful to show some of the power of the method, and that it really can be applied past standard toy examples. Suppose the utility function is changed to have a subsistence or minimum level of consumption $h$,

$$u(c) = (c-h)^{1-\gamma}.$$

Now, the optimal payoff is

$$\begin{aligned}(\hat{x}-h)^{-\gamma} &= \lambda m \\ \hat{x} &= \lambda^{-\frac{1}{\gamma}} m^{-\frac{1}{\gamma}} + h\end{aligned}$$

Evaluating the wealth constraint,

$$\begin{aligned}W_0 &= E(m\hat{x}) = \lambda^{-\frac{1}{\gamma}} E\left(m^{1-\frac{1}{\gamma}}\right) + he^{-rT} \\ \lambda^{-\frac{1}{\gamma}} &= \frac{W_0 - he^{-rT}}{E\left(m^{1-\frac{1}{\gamma}}\right)} \\ \hat{x} &= \left(W_0 - he^{-rT}\right) \frac{m^{-\frac{1}{\gamma}}}{E\left(m^{1-\frac{1}{\gamma}}\right)} + h\end{aligned}$$

The *discount factor* has not changed, so we can use the discount factor terms from the last example unchanged. In the lognormal Black-Scholes example we have been carrying along, this result gives us, corresponding to (**??**),

$$\hat{x} = \left(W_0 - he^{-rT}\right) e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T} R_T^{\alpha} + h$$

This is a very sensible answer. First and foremost, the investor guarantees the payoff $h$. Then, wealth left over after buying a bond that guarantees $h$, $\left(W_0 - he^{-rT}\right)$ is invested in the usual power utility manner. Figure 2 plots the payoffs of the optimal portfolios. You can see the left end is higher and the right end is lower. The investor sells off some performance in good states of the world to make sure his portfolio never pays off less than $h$ no matter how bad the state of the world.

## 2.2 Incomplete markets

Most of the complete markets approach goes through in incomplete markets as well. The first order condition $\hat{x} = u'^{-1}(\lambda m) - e$ still gives the optimal portfolio, but in general there are many $m$ and we don't know which one lands $\hat{x} \in \underline{X}$, the space available to the investor

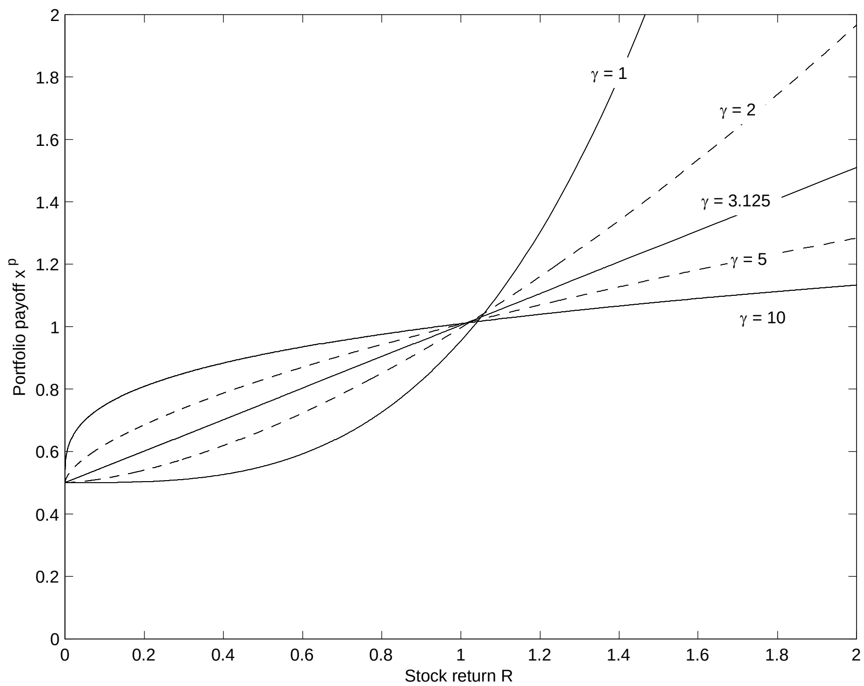


Figure 2: Portfolio problem with habit utility

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Well, what if markets are *not* complete? This is the case in the real world. Market incompleteness is also what makes portfolio theory challenging. So, let's generalize the ideas of the last section to incomplete markets.

When markets are incomplete, we have to be more careful about what actually is available to the investor. I start with a quick review of the setup and central results from Chapter 4. The payoffs available to the investor are a space $\underline{X}$. For each payoff $x \in \underline{X}$ the investor knows the price $p(x)$. Returns have price 1, excess returns have price zero. The investor can form arbitrary portfolios without short-sale constraints or transactions costs (that's another interesting extension), so the space $\underline{X}$ of payoffs is closed under linear transformations:

$$x \in \underline{X}, y \in \underline{X} \Rightarrow ax + by \in \underline{X}$$

I assume that the law of one price holds, so the price of a portfolio is the same as the price of its constituent elements.

$$p(ax + by) = ap(x) + bp(y).$$

(If not, portfolio theory would be easy and immensely profitable.)

As before, let's follow the insight that summarizing prices and payoffs with a discount factor makes the portfolio theory problem easier. From Chapter 4, we know that the law of one price implies that there is a unique discount $x^* \in \underline{X}$ such that

$$p(x) = E(x^* x) \qquad (12)$$

for all $x \in \underline{X}$. The discount factor $x^*$ is often easy to construct. For example, if the payoff space is generated as all portfolios of a finite vector of basis payoffs $\mathbf{x}$ with price vector $\mathbf{p}$, $\underline{X} = \{\mathbf{c}'\mathbf{x}\}$, then

$$x^* = \mathbf{p}'E(\mathbf{x}\mathbf{x}')^{-1}\mathbf{x}$$

satisfies $\mathbf{p} = E(x^*\mathbf{x})$ and $x^* \in \underline{X}$. Equation (9) is the continuous-time version of this equation.

If markets are complete, this is the unique discount factor. If markets are not complete, then there are many discount factors and any $m = x^* + \varepsilon$, with $E(\varepsilon x) = 0 \ \forall x \in \underline{X}$ is a discount factor. Therefore, $x^* = proj(m|\underline{X})$ for any discount factor $m$. The return corresponding to the payoff $x^*$is $R^* = x^*/p(x^*) = x^*/E(x^{*2})$. $R^*$ is the global minimum second moment return, and so it is on the lower portion of the mean-variance frontier. $x^*$ and $R^*$ need not be positive in every state of nature. Absence of arbitrage means there *exists* a positive discount factor $m = x^* + \varepsilon$, but the positive $m$ may not lie in $\underline{X}$, and there are many non-positive discount factors as well.

The canonical one-period portfolio problem is now

$$\begin{aligned} & \max_{\{\hat{x} \in \underline{X}\}} E\left[u(c)\right] \text{ s.t.} \\ c &= \hat{x} + e; \;\; W = p(\hat{x}). \end{aligned} \tag{13}$$

This is different from our first problem (1) only by the restriction $\hat{x} \in \underline{X}$: markets are incomplete, and the investor can only choose a tradeable payoff.

The first order conditions are the same as before. We can see this most transparently in the common case of a finite set of basis payoffs $\underline{X} = \{\mathbf{c}'\mathbf{x}\}$. Then, the constrained portfolio choice is $\hat{x} = \boldsymbol{\alpha}'\mathbf{x}$ and we can choose the portfolio weights $\boldsymbol{\alpha}$, respecting in this way $\hat{x} \in \underline{X}$. The portfolio problem is then

$$\max_{\{\alpha_i\}} E\left[u\left(\sum_i \alpha_i x_i + e\right)\right] \text{ s.t. } \; W = \sum_i \alpha_i p_i.$$

The first order conditions are

$$p_i \lambda = E\left[u'(\hat{x} + e)x_i\right] \tag{14}$$

for each asset $i$, where $\lambda$ is the Lagrange multiplier on the wealth constraint.

Equation (14) is our old friend $p = E(mx)$. It holds for each asset in $\underline{X}$ if and only if $u'(\hat{x}+e)/\lambda$ is a discount factor for all payoffs $\hat{x} \in \underline{X}$. We conclude that *marginal utility must be proportional to* **a** *discount factor*,

$$u'(\hat{x} + e) = \lambda m \tag{15}$$

where $m$ satisfies $p = E(mx)$ for all $x \in \underline{X}$.

We can also apply the same derivation as before, though the logic is a little trickier. We know from the law of one price that there exists an $m$ such that $p = E(mx) \forall x \in \underline{X}$, in fact there are lots of them. Thus, we can state the constraint as $W = E(m\hat{x})$ using any such $m$.

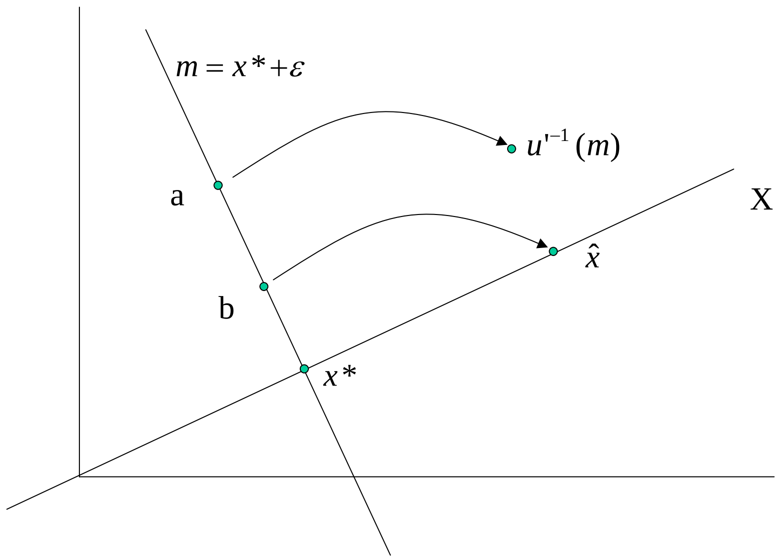


Figure 3: Portfolio problem in incomplete markets

Now the problem (13) is exactly the same as the original problem, so we can find the first order condition by choosing $\hat{x}$ in each state directly, with no mention of the original prices and payoffs.

The solution to the portfolio problem is thus once again

$$\hat{x} = u'^{-1}(\lambda m) - e.$$

If markets are complete, as above, the discount factor $m = x^*$ is unique and in $\underline{X}$. Every payoff is traded, so both $\lambda m$ and $u'^{-1}(\lambda m) - e$ are in $\underline{X}$. Hence, all we have to do is find the Lagrange multiplier to satisfy the initial wealth constraint.

If markets are not complete, we also have to pay attention to the constraint $\hat{x} \in \underline{X}$. We have derived *necessary* condition for an optimal portfolio, but not yet a *sufficient* condition. There are *many* discount factors that price assets, and for only *one* of them is the inverse marginal utility in the space of traded assets. While it's easy to construct $x^* \in \underline{X}$, for example, that may be the wrong discount factor.

Figure 3 illustrates the problem for the case $e = 0$. $\underline{X}$ is the space of traded payoffs. $x^*$ is the unique discount factor in $\underline{X}$. $m = x^* + \varepsilon$ gives the space of all discount factors. It is drawn at right angles to $\underline{X}$ since $E(\varepsilon x) = 0 \; \forall x \in \underline{X}$. The optimal portfolio $x^*$ satisfies $u'^{-1}(\lambda m) = x^*$ for some $m$. Case a shows what can go wrong if you pick the wrong $m$: $u'^{-1}(\lambda m)$ is not in the payoff space $\underline{X}$, so it can't be the optimal portfolio. Case b shows the optimal portfolio: we have chosen the right $m$ so that $u'^{-1}(\lambda m)$ is in the payoff space $\underline{X}$. $x^*$

is the wrong choice as well, since $u'^{-1}(\lambda x^*)$ takes you out of the payoff space.

As the figure suggests, markets don't *have to* be completely "complete" for $\hat{x} = u'^{-1}(\lambda x^*)$ to work. It is enough that the payoff space $\underline{X}$ is closed under (some) nonlinear transformations. If for every $x \in \underline{X}$, we also have $u'^{-1}(x) \in \underline{X}$, then $\hat{x} = u'^{-1}(\lambda x^*)$ will be tradeable, and we can again find the optimal portfolio by inverting marginal utility from the easy-to-compute unique discount factor $x^* \in \underline{X}$. A full set of options gives closure under *all* nonlinear transformations and this situation is often referred to as "complete markets," even though many shocks are not traded assets. Obviously, even less "completeness" than this can work in many applications.

What can we do? How can we pick the right $m$? In general, there are two ways to proceed. First, we can search over all possible $m$, i.e. slide up and down the $m$ hyperplane and look for the one that sends $u'^{-1}(\lambda m) \in \underline{X}$. This isn't necessarily hard, since we can set up the search as a minimization, minimizing the distance between $u'^{-1}(m)$ and $\underline{X}$. Equivalently, we can invent prices for the missing securities, solve the (now unique) complete markets problem, and search over those prices until the optimal portfolio just happens to lie in the original space $\underline{X}$. Equivalently again, we can attach Lagrange multipliers to the constraint $\hat{x} \in \underline{X}$ and find "shadow prices" that satisfy the constraints.

Second, we can start all over again by explicitly choosing portfolio weights directly in the limited set of assets at hand. This approach also leads to a lot of complexity. In addition, in most applications there are a lot more assets at hand than can conveniently be put in a maximization. For example, we often do portfolio problems with a stock and a bond, ignoring the presence of options and futures. In almost all cases of practical importance, we have to result to numerical answers, which means some approximation scheme.

Third, we can simplify or approximate the *problem*, so that $u'^{-1}(\cdot)$ is an easy function.

## 2.3 Linear-quadratic approximation and mean-variance analysis

If marginal utility is *linear*, $u'(c) = c^b - c$, then we can easily solve for portfolios in incomplete markets. I derive $\hat{x} = \hat{c}^b - \hat{e} - \left[p(\hat{c}^b) - p(\hat{e}) - W\right] R^*$, where $\hat{c}^b$ and $\hat{e}$ are mimicking payoffs for a stochastic bliss point and outside income, $W$ is initial wealth, and $R^*$ is the minimum second moment return. The portfolio gets the investor as close as possible to bliss point consumption, after hedging outside income risk, and then accepting lower consumption in the high contingent claims price states.

The problem is that marginal utility is nonlinear, while the payoff space $\underline{X}$ is only closed under linear combinations. This suggests a classic approximation: With quadratic utility, marginal utility is linear. Then we know that the inverse image of $x^* \in \underline{X}$ is also in the space of payoffs, and this is the optimal portfolio.

Analytically, suppose utility is quadratic

$$u(c) = -\frac{1}{2}(c^b - c)^2$$

where $c^b$ is a potentially stochastic bliss point. Then

$$u'(c) = c^b - c.$$

The first order condition (15) now reads

$$c^b - \hat{x} - e = \lambda m.$$

Now, we can project both sides onto the payoff space $\underline{X}$, and solve for the optimal portfolio. Since $proj(m|\underline{X}) = x^*$, this operation yields

$$\hat{x} = -\lambda x^* + proj\left(c^b - e|\underline{X}\right). \tag{16}$$

To make the result clearer, we again solve for the Lagrange multiplier $\lambda$ in terms of initial wealth. Denote by $\hat{e}$ and $\hat{c}^b$ the mimicking portfolios for preference shocks and labor income risk,

$$\begin{aligned} \hat{e} &\equiv proj\left(e|\underline{X}\right) \\ \hat{c}^b &\equiv proj\left(c^b|\underline{X}\right) \end{aligned}$$

(Projection means linear regression. These are the portfolios of asset payoffs that are closest, in mean square sense, to the labor income and bliss points.) The wealth constraint then states

$$W = p(\hat{x}) = -\lambda p(x^*) + p(\hat{c}^b) - p(\hat{e})$$

$$\frac{p(\hat{c}^b) - p(\hat{e}) - W}{p(x^*)} = \lambda$$

Thus, the optimal portfolio is

$$\hat{x} = \hat{c}^b - \hat{e} - \left[p(\hat{c}^b) - p(\hat{e}) - W\right] R^*, \tag{17}$$

where again $R^* = x^*/p(x^*) = x^*/E(x^{*2})$ is the return corresponding to the discount-factor payoff $x^*$.

The investor starts by hedging as much of his preference shock and labor income risk as possible. If these risks are traded, he will buy a portfolio that gets rid of all labor income risk $e$ and then buys bliss point consumption $c^b$. If they are not traded, he will buy a portfolio that is closest to this ideal – a mimicking portfolio for labor income and preference shock risk. Then, depending on initial wealth and hence risk aversion (risk aversion depends on wealth for quadratic utility), he invests in the minimum second moment return $R^*$. Typically (for all interesting cases) wealth is not sufficient to buy bliss point consumption, $W + p(\hat{e}) < p(\hat{c}^b)$. Therefore, the investment in $R^*$ is negative. $R^*$ is on the lower portion of the mean-variance frontier, so when you short $R^*$, you obtain a portfolio on the upper portion of the frontier.

The investment in the risky portfolio is larger (in absolute value) for lower wealth. Quadratic utility has the perverse feature that risk aversion increases with wealth to infinity at the bliss point. Given that the investor cannot buy enough assets to consume $\hat{c}^b$, $R^*$ tells him which states have the highest contingent claims prices. Obviously, sell what you have at the highest price.

In sum, *each investor's optimal portfolio is a combination of a mimicking portfolio to hedge labor income and preference shock risk, plus an investment in the (mean-variance efficient) minimum second moment return, whose size depends on risk aversion or initial wealth.*

### 2.3.1 The mean-variance frontier

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With no outside income $e = 0$, we can express the quadratic utility portfolio problem in terms of local risk aversion,

$$\hat{R} = R^f + \frac{1}{\gamma}\left(R^f - R^*\right).$$

This expression makes it clear that the investor holds a mean-variance efficient portfolio, further away from the risk free rate as risk aversion declines.

---

Traditional mean-variance analysis focuses on a special case: the investor has no job, so labor income is zero, the bliss point is nonstochastic, and a riskfree rate is traded. This special case leads to a very simple characterization of the optimal portfolio. Equation (17) specializes to

$$\hat{x} = c^b - \left(\frac{c^b}{R^f} - W\right) R^* \tag{18}$$

$$\hat{R} = \frac{\hat{x}}{W} = R^* + \frac{c^b}{R^f W}\left(R^f - R^*\right) \tag{19}$$

In Chapter 5, we showed that the mean-variance frontier is composed of all portfolios of the form $R^* + \alpha(R^f - R^*)$. Therefore, *investors with quadratic utility and no labor income all hold mean-variance efficient portfolios.* As $W$ rises or $c^b$ declines, the investor becomes more risk averse. When $W$ can finance bliss-point consumption for sure, $WR^f = c^b$, the investor becomes infinitely risk averse and holds only the riskfree rate $R^f$.

Obviously, these global implications – rising risk aversion with wealth – are perverse features of quadratic utility, which should be thought of as a local approximation. For this reason, it is interesting and useful to express the portfolio decision in terms of the local risk aversion coefficient.

Write (19) as

$$\hat{R} = R^f + \left(\frac{c^b}{R^f W} - 1\right)\left(R^f - R^*\right) \tag{20}$$

Local risk aversion for quadratic utility is

$$\gamma = -\frac{cu''(c)}{u'(c)} = \frac{c}{c^b - c} = \left(\frac{c^b}{c} - 1\right)^{-1}.$$

Now we can write the optimal portfolio

$$\hat{R} = R^f + \frac{1}{\gamma}\left(R^f - R^*\right). \tag{21}$$

where we evaluate local risk aversion $\gamma$ at the point $c = R^f W$.

The investor invests in a mean-variance efficient portfolio, with larger investment in the risky asset the lower his risk aversion coefficient. Again, $R^*$ is on the lower part of the mean-variance frontier, thus a short position in $R^*$ generates portfolios on the upper portion of the frontier. $R^f W$ is the amount of consumption the investor would have in period 2 if he invested everything in the risk free asset. This is the sensible place at which to evaluate risk aversion. For example, if you had enough wealth to buy bliss point consumption $R^f W = c^b$, you would do it and be infinitely risk averse.

### 2.3.2 Formulas

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I evaluate the mean-variance formula $\hat{R} = R^f + \frac{1}{\gamma}\left(R^f - R^*\right)$ for the common case of a riskfree rate $R^f$ and vector of excess returns $R^e$ with mean $\mu$ and covariance matrix $\Sigma$. The result is

$$R^f - R^* = \left(\frac{R^f}{1 + \mu'\Sigma^{-1}\mu}\right)\mu'\Sigma^{-1}R^e$$

The terms are familiar from simple mean-variance maximization: finding the mean-variance frontier directly we find that mean-variance efficient weights are all of the form $w = \lambda\mu'\Sigma^{-1}$ and the maximum Sharpe ratio is $\mu'\Sigma^{-1}\mu$.

---

Formula (21) is a little dry, so it's worth evaluating a common instance. Suppose the payoff space consists of a riskfree rate $R^f$ and $N$ assets with excess returns $R^e$, so that portfolio returns are all of the form $R^p = R^f + w'R^e$. Denote $\mu = E(R^e)$ and $\Sigma = cov(R^e)$. Let's find $R^*$ and hence (21) in this environment.

Repeating briefly the analysis of Chapter 5, we can find

$$x^* = \frac{1}{R^f} - \frac{1}{R^f}\mu'\Sigma^{-1}(R^e - \mu).$$

(Check that $x^* \in \underline{X}$, $E(x^* R^f) = 1$ and $E(x^* R^e) = 0$, or derive it from $x^* = \alpha R^f + w'\left[R^e - \mu\right]$.) Then

$$p(x^*) = E(x^{*2}) = \frac{1}{R^{f2}} + \frac{1}{R^{f2}}\mu'\Sigma^{-1}\mu$$

so

$$R^* = \frac{x^*}{E(x^{*2})} = R^f \frac{1 - \mu'\Sigma^{-1}(R^e - \mu)}{1 + \mu'\Sigma^{-1}\mu}$$

and

$$\begin{aligned} R^f - R^* &= R^f - \frac{R^f}{1 + \mu'\Sigma^{-1}\mu} + \frac{R^f}{1 + \mu'\Sigma^{-1}\mu}\mu'\Sigma^{-1}(R^e - \mu) \\ R^f - R^* &= \frac{R^f}{1 + \mu'\Sigma^{-1}\mu}\mu'\Sigma^{-1}R^e \end{aligned} \tag{22}$$

To give a reference for these formulas, consider the standard approach to finding the mean-variance frontier. Let $R^{ep}$ be the excess return on a portfolio. Then we want to find

$$\begin{aligned} \min \sigma^2(R^{ep}) \text{ s.t. } E(R^{ep}) &= E \\ \min_{\{w\}} w'\Sigma w \text{ s.t. } w'\mu &= E \end{aligned}$$

The first order conditions give

$$w = \lambda \Sigma^{-1}\mu$$

where $\lambda$ scales up and down the investment to give larger or smaller mean. Thus, the portfolios on the mean-variance frontier have excess returns of the form

$$R^{ep} = \lambda \mu'\Sigma^{-1}R^e$$

This is a great formula to remember: $\mu'\Sigma^{-1}$ gives the weights for a mean-variance efficient investment. You can see that (22) is of this form.

The Sharpe ratio or slope of the mean-variance frontier is

$$\frac{E(R^{ep})}{\sigma(R^{ep})} = \frac{\mu'\Sigma^{-1}\mu}{\sqrt{\mu'\Sigma^{-1}\mu}} = \sqrt{\mu'\Sigma^{-1}\mu}$$

Thus, you can see that the term scaling $R^f - R^*$ scales with the market Sharpe ratio.

We could of course generate the mean-variance frontier from the risk free rate and any efficient return. For example, just using $\mu'\Sigma^{-1}R^e$ might seem simpler than using (22), and it is simpler when making computations. The *particular* mean-variance efficient portfolio $R^f - R^*$ in (22) has the delicious property that it is the optimal portfolio for risk aversion equal to one, and the units of any investment have directly the interpretation as a risk aversion coefficient.

### 2.3.3 The market portfolio and two-fund theorem

In a market of quadratic utility, $e = 0$ investors, we can aggregate across people and express the optimal portfolio as

$$\hat{R}^i = R^f + \frac{\gamma^m}{\gamma^i}\left(R^m - R^f\right)$$

This is a "two-fund" theorem – the optimal portfolio for every investor is a combination of the risk free rate and the market portfolio. Investors hold more or less of the market portfolio according to whether they are less or more risk averse than the average investor.

---

I have used $R^*$ so far as the risky portfolio. If you read Chapter 5, this will be natural. However, conventional mean-variance analysis uses the "market portfolio" on the top of the mean variance frontier as the reference risky return. It's worth developing this representation and the intuition that goes with it.

Write the portfolio choice of individual $i$ from (20) as

$$\hat{R}^i = R^f + \frac{1}{\gamma^i}\left(R^f - R^*\right). \tag{23}$$

The market portfolio $\hat{R}^m$ is the wealth-weighted average of individual portfolios, or the return on the sum of individual payoffs,

$$\hat{R}^m \equiv \frac{\sum_{i=1}^N \hat{x}^i}{\sum_{j=1}^N W^j} = \frac{\sum_{i=1}^N W^i \hat{R}^i}{\sum_{j=1}^N W^j}$$

Summing (**??**) over individuals, then,

$$\hat{R}^m = R^f + \frac{\sum_{i=1}^N W^i \frac{1}{\gamma^i}}{\sum_{j=1}^N W^j}\left(R^f - R^*\right).$$

We can define an "average risk aversion coefficient" as the wealth-weighted average of (inverse) risk aversion coefficients[1],

$$\frac{1}{\gamma^m} \equiv \frac{\sum_{i=1}^N W^i \frac{1}{\gamma^i}}{\sum_{j=1}^N W^j}$$

so

$$\hat{R}^m = R^f + \frac{1}{\gamma^m}\left(R^f - R^*\right).$$

Using this relation to substitute $R^m - R^f$ in place of $R^f - R^*$ in (23), we obtain

$$\hat{R}^i = R^f + \frac{\gamma^m}{\gamma^i}\left(R^m - R^f\right) \tag{24}$$

*The optimal portfolio is split between the risk free rate and the market portfolio. The weight on the market portfolio return depends on individual relative to average risk aversion.*

[1] "Market risk aversion" is also the local risk aversion of an investor with the average blisspoint and average wealth,

$$\frac{1}{\gamma^m} = \frac{\frac{1}{N}\sum_{i=1}^N c^{bi}}{R^f \frac{1}{N}\sum_{i=1}^N W^i} - 1.$$

The "market portfolio" here is the average of *all* assets held. If there are bonds in "net supply" then they are included in the market portfolio, and the remaining riskfree rate is in "zero net supply." Since $x^i = c^i$, the market portfolio is also the claim to total consumption.

Since any two mean-variance efficient portfolios span the frontier, $R^m$ and $R^f$ for example, we see that optimal portfolios follow a *two-fund theorem*. This is very famous in the history of finance. It was once taken for granted that each individual needed a tailored portfolio, riskier stocks for less risk averse investors. Investment companies still advertise how well they listen. In this theory, the only way people differ is by their risk aversion, so all investors' portfolios can be provided by two funds, a "market portfolio" and a risk free security.

This is all illustrated in the classic mean-variance frontier diagram

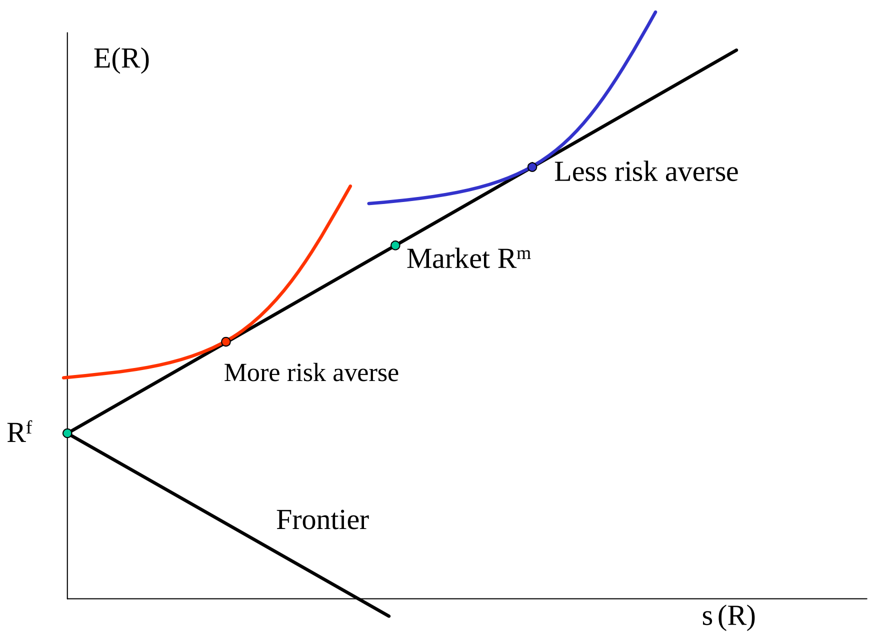


Figure 4: Mean-variance efficient portfolios.

## 2.4 Nontradeable income

I introduce two ways of expressing mean-variance portfolio theory with outside income. First, the *overall* portfolio, including the hedge portfolio for outside income, is still on the mean-variance frontier. Thus, we could use classic analysis to determine the right overall portfolio – keeping in mind that the overall market portfolio includes hedge portfolios for the average investors outside income too – and then subtract off the hedge portfolio for individual outside income in order to arrive at the individual's asset portfolio. Second, we can express the individual's portfolio as 1) the market *asset* portfolio, adjusted for risk aversion and the composition of wealth, 2) the average outside income hedge portfolio for all other investors, adjusted again for risk aversion and wealth and finally 3) the hedge portfolio for the individual's idiosyncratic outside income.

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The mean-variance frontier is a beautiful and classic result, but most investors do in fact have jobs, business income or real estate. Here, I attempt some restatments of the more interesting case with labor income and preference shocks to bring them closer to mean-variance intuition.

One way to do this is to think of labor or business income as part of a "total portfolio". Then, the total portfolio *is* still mean-variance efficient, but we have to adjust the asset portfolio for the presence of outside income.

To keep it simple, keep a nonstochastic bliss point, $c^b$. Then, equation (17) becomes

$$\hat{x} = c^b - \hat{e} - \left[p(c^b) - p(\hat{e}) - W\right] R^*$$

We can rewrite this as

$$\hat{e} + \hat{x} = c^b - \left[p(c^b) - (W + p(\hat{e}))\right] R^*$$

The left hand side is the "total payoff", consisting of the asset payoff $\hat{x}$ and the labor income hedge portfolio $\hat{e}$ (Consumption is this payoff plus residual labor income, $c = \hat{x} + e = \hat{x} + (e - \hat{e}) + \hat{e}$.)

We define a rate of return on the "total portfolio" as the total payoff – asset portfolio plus human capital – divided by total value, and proceed as before,

$$\begin{aligned}
\hat{R}^{tp} &= \frac{\hat{e} + \hat{x}}{W + p(\hat{e})} = \frac{c^b}{W + p(\hat{e})} - \left[\frac{c^b}{R^f\left[W + p(\hat{e})\right]} - 1\right] R^* \\
&= R^* + \frac{c^b}{R^f\left[W + p(\hat{e})\right]}\left(R^f - R^*\right) \\
\hat{R}^{tp} &= R^* + \frac{1}{\gamma}\left(R^f - R^*\right)
\end{aligned}$$

Now $\gamma$ is defined as the local risk aversion coefficient given $c^b$ and using the value of initial wealth and the tradeable portfolio closest to labor income, invested at the risk free rate. Thus, we can say that the *total portfolio is mean-variance efficient.* We can also aggregate just as before, to express

$$\hat{R}^{tp,i} = R^f + \frac{\gamma^m}{\gamma^i}\left(R^{tp,m} - R^f\right) \tag{25}$$

where $R^m$ is now the total wealth portfolio *including* the outside income portfolios,

This representation makes it seem like nothing much has changed. However the *asset* portfolio – the thing the investor actually buys – changes dramatically. $\hat{e}$ is a payoff the investor already owns. Thus, to figure out the *asset* market payoff, you have to *subtract* the labor income hedge portfolio from the appropriate mean-variance efficient portfolio.

$$\hat{R}^i = \frac{\hat{x}^i}{W^i} = \frac{p(\hat{e}^i)+W^i}{W^i}\left(\frac{\hat{e}^i+\hat{x}^i}{p(\hat{e}^i)+W^i} - \frac{\hat{e}^i}{p(\hat{e}^i)+W^i}\right) \tag{26}$$

$$= \left(1+\frac{p(\hat{e}^i)}{W^i}\right)\hat{R}^{tp,i} - \left(\frac{p(\hat{e}^i)}{W^i}\right)\frac{\hat{e}^i}{p(\hat{e}^i)} \tag{27}$$

$$= \left(1+\frac{p(\hat{e}^i)}{W^i}\right)\hat{R}^{tp,i} - \left(\frac{p(\hat{e}^i)}{W^i}\right)\hat{R}^{e,i} \tag{28}$$

This can be a large correction. Also, in this representation the "market portfolio" $\hat{R}^{tp}$ includes everyone else's hedge portfolio. It is *not* the average of actual asset market portfolios.

For that reason, a slightly more complex representation is also useful. We can break up the "total" return to the two components, a "hedge portfolio return" and the asset portfolio return,

$$\begin{aligned} R^{tp} &= \frac{\hat{e}^i+\hat{x}^i}{p(\hat{e}^i)+W^i} \\ &= \frac{p(\hat{e}^i)}{p(\hat{e}^i)+W^i}\frac{\hat{e}^i}{p(\hat{e}^i)} + \frac{W^i}{p(\hat{e}^i)+W^i}\frac{\hat{x}^i}{W^i} \\ &= (1-w^i)\hat{R}^{e,i} + w^i\hat{R}^i \end{aligned}$$

Here

$$\hat{R}^{e,i} = \frac{\hat{e}^i}{p(\hat{e}^i)};\ w^i = \frac{W^i}{p(\hat{e}^i)+W^i}; 1-w^i = \frac{p(\hat{e}^i)}{p(\hat{e}^i)+W^i}.$$

The same decomposition works for $R^{tp,m}$. Then, substituting in (25),

$$(1-w^i)\hat{R}^{e,i} + w^i\hat{R}^i = R^f + \frac{\gamma^m}{\gamma^i}\left((1-w^m)\,\hat{R}^{e,m} + w^m\hat{R}^m - R^f\right)$$

and hence

$$\hat{R}^i - R^f = \frac{\gamma^m}{\gamma^i}\frac{w^m}{w^i}\left(\hat{R}^m - R^f\right) + \frac{\gamma^m}{\gamma^i}\frac{w^m}{w^i}\frac{(1-w^m)}{w^m}\left(\hat{R}^{e,m} - R^f\right) - \frac{(1-w^i)}{w^i}\left(\hat{R}^{e,i} - R^f\right) \tag{29}$$

This representation emphasizes a deep point, you only deviate from the market portfolio to the extent that you are *different* from everyone else. The first term says that an individual's actual portfolio scales up or down the market portfolio according to the individual's risk aversion and the relative weight of asset wealth in total wealth. If you have more outside wealth relative to total, $w^i$ is lower, you hold a less risk averse position in your *asset* portfolio. The second term is the hedge portfolio for the average investor's labor income The next terms describe how you should change your portfolio if the character of your outside income is *different* from everyone else's.

A few examples will clarify the formula. First, suppose that outside income is nonstochastic, so $R^e = R^f$. The second terms vanish, and we are left with

$$\hat{R}^i - R^f = \frac{\gamma^m}{\gamma^i}\frac{w^m}{w^i}\left(\hat{R}^m - R^f\right)$$

This is the usual formula except that risk aversion is not multiplied by the share of asset wealth in total wealth,

$$\gamma^i w^i = \gamma^i \frac{W^i}{W^i + p(e^i)}.$$

An individual with a lot of outside income $p(e^i)$ is sitting on a bond. Therefore, his asset market portfolio should be shifted towards risky assets; his asset market portfolio is the same as that of an investor with no outside income but a lot less risk aversion. This explains why "effective risk aversion" for the asset market portfolio is in (29) multiplied by wealth.

Second, suppose that the investor has the same wealth and relative wealth as the market, $\gamma^i = \gamma^m$ and $w^i = w^m$, but outside income is stochastic. Then expression (29) simplifies to

$$\hat{R}^i - R^f = \left(\hat{R}^m - R^f\right) + \frac{(1-w)}{w}\left[\hat{R}^{e,m} - \hat{R}^{e,i}\right]$$

This investor hold the market portfolio (this time the actual, traded-asset market portfolio), plus a hedge portfolio derived from the *difference* between his income and the average investor's income. If the investor is just like the average investor in this respect as well, then he just holds the market portfolio of traded assets. But suppose this investor's outside income is a bond, $\hat{R}^{ej} = R^f$, while the average investor has a stochastic outside income. Then, the investor's *asset* portfolio will include the hedge portfolio for aggregate outside income. He will do better in a mean-variance sense by providing this "outside income insurance" to the average investor.

The market portfolio $\hat{R}^m$ is *not* on the mean-variance frontier anymore. The "total" market portfolio $\hat{R}^{tp,m} = w^m \hat{R}^m + (1 - w^m)\hat{R}^{e,m}$ *is* on the mean-variance frontier, but that includes the average investor's hedge portfolio for outside income. Thus, the presence of an average level of outside income justifies multifactor models such as the Famous Fama French three Factor model, if the additional factors are mimicking portfolios for human capital risks.

# 3 Choosing payoffs in intertemporal, dynamic problems

One-period problems are fun and pedagogically attractive, but not realistic. People live a long time. One-period problems would still be a useful guide if the world were i.i.d., so that each day looked like the last. Alas, the overwhelming evidence from empirical work is that the world is *not* i.i.d. Expected returns, variances and covariances all change through time. Even if this were not the case, individual investors' outside incomes vary with time, age and the lifecycle. We need a portfolio theory that incorporates long-lived agents, and allows for time-varying moments of asset returns. Furthermore, many dynamic setups give rise to incomplete markets, since shocks to forecasting variables are not traded.

This seems like a lot of complexity, and it is. Fortunately, with a little reinterpretation of symbols, we can apply everything we have done for one-period markets to this intertemporal dynamic world.

I start with a few classic examples that should be in every financial economists' toolkit, and then draw the general point.

## 3.1 Portfolios and discount factors in intertemporal models

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The identical optimal portfolio formulas hold in an intertemporal model,

$$\beta^t u'(\hat{x}_t + e_t) = \lambda m_t \tag{30}$$

$$\hat{x}_t = u'^{-1}(\lambda m_t/\beta^t) - e_t. \tag{31}$$

where we now interpret $\hat{x}_t$ to be the flow of dividends (payouts) of the optimal portfolio, and $e_t$ is the flow of outside income.

---

Start with an investor with no outside income; his utility function is

$$E\sum_{t=1}^{\infty}\beta^t u(c_t).$$

He has initial wealth $W$ and he has a *stream* of outside income $\{e_t\}$. His problem is to pick a *stream* of payoffs or dividends $\{\hat{x}_t\}$, which he will eat, $c_t = \hat{x}_t + e_t$.

As before, we summarize the assets available to the investor by a discount factor $m$. Thus, the problem is

$$\max_{\{\hat{x}_t \in \underline{X}\}} E\sum_{t=1}^{\infty}\beta^t u(\hat{x}_t + e_t) \ s.t. \ W = E\sum_{t=1}^{T} m_t \hat{x}_t$$

Here $m_t$ represents a discount factor *process*, i.e. for every payoff $x_t$, $m_t$ generates prices $p$ by

$$p = E\sum_{t=1}^{\infty} m_t x_t.$$

As before, absence of arbitrage and the law of one price guarantee that we can represent the prices and payoffs facing the investor by such a discount factor process.

The first order conditions to this problem are ($\partial/\partial\hat{x}_{it}$ in state $i$ at time $t$)

$$\beta^t u'(\hat{x}_t + e_t) = \lambda m_t \tag{32}$$

Thus, once again the optimal payoff is characterized by

$$\hat{x}_t = u'^{-1}(\lambda m_t/\beta^t) - e_t. \tag{33}$$

The formula is only different because utility of consumption at time t is multiplied by $\beta^t$. If $m$ is unique (complete markets), then we are done. If not, then again we have to choose the right $\{m_t\}$ so that $\{\hat{x}\} \in \underline{X}$. (We have to think in some more detail what this payoff space looks like when markets are not complete.)

As before, this condition characterizes the solution up to initial wealth. To match it to a specific initial wealth (or to find what wealth corresponds to a choice of $\lambda$), we impose the constraint,

$$E\sum_t m_t u'^{-1}(\lambda m_t) = W.$$

The corresponding continuous time formulation is

$$\max E \int_{t=0}^{\infty} e^{-\rho t} u(\hat{x}_t + e_t)dt \quad \text{s.t.} \;\; W = E\int_{t=0}^{\infty} m_t \hat{x}_t dt$$

giving rise to the identical conditions

$$e^{-\rho t} u'(\hat{x}_t + e_t) = \lambda m_t \tag{34}$$

$$\hat{x}_t = u'^{-1}(\lambda m_t / e^{-\rho t}) - e_t. \tag{35}$$

## 3.2 The power-lognormal problem.

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We solve for the optimal infinite-horizon portfolio problem in the lognormal iid setup. The answer is that optimal consumption or dividend is a power function of the current stock value,

$$\hat{x}_t = (\text{const.}) \;\times \left(\frac{S_t}{S_0}\right)^{\alpha}; \;\; \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}$$

---

To see this analysis more concretely, and for its own interest, let's solve a classic problem. The investor has no outside income, lives forever and wants intermediate consumption, and has power utility

$$\max E \int_{t=0}^{\infty} \frac{\hat{x}_t^{1-\gamma}}{1-\gamma} dt.$$

He can dynamically trade, resulting in "complete" markets.

Once we have a discount factor $m_t$ that represents asset markets, the answer is simple. From (35)

$$\hat{x}_t = \lambda^{-\frac{1}{\gamma}} \left(e^{\rho t} m_t\right)^{-\frac{1}{\gamma}}$$

As before, we can solve for $\lambda$,

$$W = E\int_{t=0}^{\infty} m_t \lambda^{-\frac{1}{\gamma}} \left(e^{\rho t} m_t\right)^{-\frac{1}{\gamma}} dt$$
$$W = \lambda^{-\frac{1}{\gamma}} \int_{t=0}^{\infty} e^{-\frac{\rho}{\gamma}t}\, m_t^{1-\frac{1}{\gamma}}\, dt$$

so the optimal payoff is

$$\frac{\hat{x}_t}{W} = \frac{e^{-\frac{\rho}{\gamma}t}\, m_t^{-\frac{1}{\gamma}}}{\int_{t=0}^{\infty} e^{-\frac{\rho}{\gamma}t}\, m_t^{1-\frac{1}{\gamma}}\, dt} \tag{36}$$

The analogy to the one-period result (6) is strong. However, the "return" is now a dividend at time $t$ divided by an initial value, an insight I follow up on below.

We might insist that the problem be *stated* in terms of a discount factor. But in practical problems, we will first face the technical job of find the discount factor that represents a given set of asset prices and payoffs, so to make the analysis concrete and to solve a classic problem, let's introduce some assets and find their discount factor. As before, a stock and bond follow

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{37}$$
$$\frac{dB}{B} = r dt. \tag{38}$$

(Think of $S$ and $B$ as the cumulative value process with dividends reinvested, if you're worried about transversality conditions. What matters is a stock return $dR = \mu dt + \sigma dz$ and bond return $rdt$.) This is the same setup as the iid lognormal environment of section 2.1.1, but the investor lives forever and values intermediate consumption rather than living for one period and valuing terminal wealth.

Fortunately, we've already found the discount factor, both in chapter 17 and in equation (9) above, $m_t = \Lambda_t/\Lambda_0$ where

$$\frac{d\Lambda}{\Lambda} = -r dt - \frac{\mu - r}{\sigma} dz. \tag{39}$$

We can substitute $d\ln S$ for $dz$ and solve (37)-(**??**), (algebra below) resulting in

$$\frac{\Lambda_t}{\Lambda_0} = e^{\frac{1}{2}\left(\frac{\mu-r}{\sigma^2}-1\right)(\mu+r)t} \times \left(\frac{S_t}{S_0}\right)^{-\frac{\mu-r}{\sigma^2}}.$$

And thus, for power utility, (36) becomes

$$\hat{x}_t = (\text{const.}) \times \left(\frac{S_t}{S_0}\right)^{\alpha}$$

where again

$$\alpha = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}$$

*Optimal consumption at date t is a power function of the stock value at that date.* As you can guess, and as I'll show below, one way to implement this rule is to invest a constantly rebalanced fraction of wealth $\alpha$ in stocks, and to consume a constant fraction of wealth as well. But this is a complete market, so there are lots of equivalent ways to implement this rule.

Evaluating the constant – the denominator of (36) takes a little more algebra and is not very revealing, but here is the final answer:

$$\frac{\hat{x}_t}{W} = \frac{1}{\gamma}\left[\rho + (\gamma-1)\left(r + \frac{1}{2}\gamma\alpha^2\sigma^2\right)\right] e^{-\frac{1}{\gamma}\left[\rho+\frac{1}{2}(\gamma\alpha-1)(\mu+r)\right]t}\left(\frac{S_t}{S_0}\right)^{\alpha} \tag{40}$$

*Algebra:*

$$\begin{aligned} d\ln\Lambda &= \frac{d\Lambda}{\Lambda} - \frac{1}{2}\frac{d\Lambda^2}{\Lambda^2} = -\left[r + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right]dt - \frac{\mu-r}{\sigma}dz \\ d\ln S &= \frac{dS}{S} - \frac{1}{2}\frac{dS^2}{S^2} = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dz \end{aligned}$$

For the numerator, we want to express the answer in terms of $S_t$. Substituting $d\ln S$ for $dz$,

$$\begin{aligned} d\ln\Lambda &= -\left[r + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right]dt - \frac{\mu-r}{\sigma^2}\left[d\ln S - \left(\mu - \frac{1}{2}\sigma^2\right)dt\right] \\ d\ln\Lambda &= \left[-r - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 + \frac{\mu(\mu-r)}{\sigma^2} - \frac{1}{2}(\mu-r)\right]dt - \frac{\mu-r}{\sigma^2}d\ln S \\ d\ln\Lambda &= \left[-r - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 + \frac{\mu(\mu-r)}{\sigma^2} - \frac{1}{2}(\mu-r)\right]dt - \frac{\mu-r}{\sigma^2}d\ln S \\ d\ln\Lambda &= \frac{1}{2}\left[\frac{\mu-r}{\sigma^2} - 1\right](\mu+r)dt - \frac{\mu-r}{\sigma^2}d\ln S \\ \ln\Lambda_t - \ln\Lambda_0 &= \frac{1}{2}\left[\frac{\mu-r}{\sigma^2} - 1\right](\mu+r)t - \frac{\mu-r}{\sigma^2}\left(\ln S_t - \ln S_0\right) \end{aligned}$$

$$m_t = \frac{\Lambda_t}{\Lambda_0} = e^{\frac{1}{2}\left(\frac{\mu-r}{\sigma^2}-1\right)(\mu+r)t}\left(\frac{S_t}{S_0}\right)^{-\frac{\mu-r}{\sigma^2}}.$$

$$\begin{aligned} e^{-\frac{\rho}{\gamma}t}\, m_t^{-\frac{1}{\gamma}} &= e^{-\frac{\rho}{\gamma}t - \frac{1}{\gamma}\frac{1}{2}\left(\frac{\mu-r}{\sigma^2}-1\right)(\mu+r)t}\left(\frac{S_t}{S_0}\right)^{\frac{\mu-r}{\gamma\sigma^2}} \\ &= e^{-\frac{1}{\gamma}\left[\rho+\frac{1}{2}\left(\frac{\mu-r}{\sigma^2}-1\right)(\mu+r)\right]t}\left(\frac{S_t}{S_0}\right)^{\frac{\mu-r}{\gamma\sigma^2}} \end{aligned}$$

For the denominator, it's easier to express $\Lambda$ in terms of a normal random variable.

$$
\begin{aligned}
\ln \Lambda_t - \ln \Lambda_0 &= -\left[r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right] t - \frac{\mu - r}{\sigma}\sqrt{t}\varepsilon \\
m_t^{1-\frac{1}{\gamma}} &= e^{-\left(1-\frac{1}{\gamma}\right)\left[r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right]t-\frac{\mu-r}{\sigma}\left(1-\frac{1}{\gamma}\right)\sqrt{t}\varepsilon} \\
E\left(m_t^{1-\frac{1}{\gamma}}\right) &= e^{\left\{-\left(1-\frac{1}{\gamma}\right)\left[r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right]+\frac{1}{2}\left[\frac{\mu-r}{\sigma}\left(1-\frac{1}{\gamma}\right)\right]^2\right\}t} \\
&= e^{-\left(1-\frac{1}{\gamma}\right)\left\{r+\frac{1}{2}\frac{1}{\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right\}t}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty e^{-\frac{\rho}{\gamma}t} E\left(m_t^{1-\frac{1}{\gamma}}\right) dt &= \int_0^\infty e^{-\frac{\rho}{\gamma}t} e^{-\left(1-\frac{1}{\gamma}\right)\left\{r+\frac{1}{2}\frac{1}{\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right\}t} dt \\
&= \int_0^\infty e^{-\frac{1}{\gamma}\left\{\rho+(\gamma-1)\left[r+\frac{1}{2}\frac{1}{\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]\right\}t} dt \\
&= \frac{\gamma}{\rho + (\gamma - 1)\left[r + \frac{1}{2}\frac{1}{\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\hat{x}_t}{W} &= \frac{\rho + (\gamma - 1)\left[r + \frac{1}{2}\frac{1}{\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]}{\gamma} e^{-\frac{1}{\gamma}\left[\rho+\frac{1}{2}\left(\frac{\mu-r}{\sigma^2}-1\right)(\mu+r)\right]t}\left(\frac{S_t}{S_0}\right)^{\frac{\mu-r}{\gamma\sigma^2}} \\
\frac{\hat{x}_t}{W} &= \frac{1}{\gamma}\left(\rho + (\gamma - 1)\left[r + \frac{1}{2}\gamma\alpha^2\sigma^2\right]\right) e^{-\frac{1}{\gamma}\left[\rho+\frac{1}{2}(\gamma\alpha-1)(\mu+r)\right]t}\left(\frac{S_t}{S_0}\right)^{\alpha}.
\end{aligned}
$$

## 3.3 A mapping to one-period problems

The analogy in the above examples to the one-period analysis is striking. Obviously, one-period and multiperiod models are the same in a deep sense.

To make the analogy closest, let us define an expectation operator that adds over *time* using $\beta^t$ or $e^{-\rho t}$ as it adds over *states* using probabilities. Thus, define

$$
\begin{aligned}
\text{one period:} \qquad & \mathcal{E}(x) \equiv E(x_1) = \sum_s \pi(s) x_1(s) \\
\text{infinite period, discrete:} \qquad & \mathcal{E}(x) \equiv E\sum_{t=1}^{\infty}\beta^t x_t = \sum_{t=1}^{\infty}\sum_{s_t}\beta^t\pi(s_t)x_t(s_t) \\
\text{infinite period, continuous} \quad : \quad & \mathcal{E}(x) \equiv E\int_0^\infty e^{-\rho t} x_t dt
\end{aligned}
$$

It is convenient to take $\beta$ as the investor's discount factor, but not necessary.

With this definition, infinite horizon portfolio theory looks *exactly* like one period theory. We write asset pricing as

$$
p(x) = E\sum_{t=1}^{\infty}\beta^t m_t x_t = \mathcal{E}(mx).
$$

Here it is convenient to start with a discount factor that is scaled by $\beta^t$ in order to then multiply by $\beta^t$. In the canonical example which was expressed $m_t = \beta^t u\prime(c_t)/u'(c_0)$ we now have $m_t = u\prime(c_t)/u'(c_0)$. This has the advantage that $m_t$ is stationary.

(One problem with this definition is that the weights over time do not add up to one, $\mathcal{E}(1) = \beta/(1-\beta)$. One can define $\mathcal{E}(x) = \frac{(1-\beta)}{\beta}\sum \beta^t E(x_t)$ to restore this property, but then we must write pricing as $p(x) = \beta/(1-\beta)\mathcal{E}(mx)$. I choose the simpler pricing equation, at the cost that you have to be careful when taking long run means $\mathcal{E}$ of constants.)

The investor's objective is

$$\begin{aligned} \max E \sum_t \beta^t u(c_t) &= \max \mathcal{E}\left[u(c)\right] \\ c_t &= \hat{x}_t + e_t \end{aligned}$$

The constraint is

$$W = E\sum_t \beta^t m_t \hat{x}_t = \mathcal{E}(m\hat{x})$$

In sum, we are *exactly* back to

$$\max \mathcal{E}\left[u(\hat{x}_t + e_t)\right] \text{ s.t. } W = \mathcal{E}(m\hat{x})$$

The first order conditions are

$$\begin{aligned} u'(\hat{x}+e) &= \lambda m \\ \hat{x} &= u'^{-1}(\lambda m) - e \end{aligned}$$

*exactly* as before. (We rescaled $m$, which is why it's not $m/\beta^t$ as in (33).)

With power utility and no outside income, we can evaluate the constraint as

$$W = \mathcal{E}(m\lambda^{-\frac{1}{\gamma}} m^{\frac{1}{\gamma}})$$

so again the complete problem is

$$\frac{\hat{x}_t}{W} = \frac{m_t^{-\frac{1}{\gamma}}}{\mathcal{E}(m^{1-\frac{1}{\gamma}})}$$

All the previous analysis goes through unchanged!

*Units*

We do, however, have to reinterpret the symbols.$\hat{x}/W$ is now a *dividend stream* divided by its time-0 price. This, apparently is the right generalization of "return" to an infinite-horizon model. More generally, for any payoff stream I think it is better to call the "return" a "yield,"

$$y_t = \frac{x_t}{p(\{x_t\})} = \frac{x_t}{\mathcal{E}(mx)}.$$

Its typical size will be something like 0.04 not 1.04. Similarly, we can define"excess yields", which are the zero-price objects as

$$y_t^e = y_t^1 - y_t^2.$$

The *risk free* payoff is thus one in all states and dates, a perpetuity

$$x_t^f = 1.$$

*risk free yield* is therefore

$$y_t^f = \frac{1}{p(\{1\})}$$

This is, in fact, the coupon yield of the perpetuity.

I think this observation alone makes a good case for looking at prices and payoff streams rather than one period returns. In a Merton or period to period analysis, a long term bond is a security that is attractive because its price happens to go up a lot when interest rates decline. Thus, it provides a good hedge for a long-term highly risk averse investor. The fact that a 10 year bond is the riskless asset for an investor with a 10 year horizon, or an indexed perpetuity is the riskless asset for an investor with an infinite horizon, is a feature hidden deep in value functions. But once you look at prices and payoffs, it's just obvious that the indexed perpetuity is the riskless asset for a long-term investor.

Thus, in place of our usual portfolios and payoff spaces, we have spaces of yields,

$$\underline{Y} \equiv \{y \in \underline{X} : p(y) = 1\},$$

$$\underline{Y}^e \equiv \{y^e \in \underline{X} : p(y^e) = 0\}.$$

It's natural to define a *long-run mean / long-run variance frontier* which solves

$$\min_{\{y \in \underline{Y}\}} \ \mathcal{E}(y^2) \ \text{ s.t. } \mathcal{E}(y) = \mu.$$

"Long run variance" prizes stability over *time* as well as stability across states of nature. If we redo exactly the same algebra as before, we find that the long-run frontier is generated as

$$y^{mv} = y^* + wy^{e*}. \tag{41}$$

Here, $y^*$ is the discount-factor mimicking portfolio return,

$$y^* = \frac{x^*}{p(x^*)} = \frac{x^*}{\mathcal{E}(x^{*2})}. \tag{42}$$

If a riskfree rate is traded, $y^{e*}$ is simply

$$y^{e*} = \frac{y^f - y^*}{y^f}. \tag{43}$$

The mean-variance frontier of excess returns is

$$\min_{\{y^e \in \underline{Y}^e\}} \ \mathcal{E}(y^{e2}) \ \text{ s.t. } \mathcal{E}(y^e) = \mu.$$

This frontier is generated simply by

$$y^{emv} = wy^{e*} \ w \in \Re$$

*Incomplete markets and the long-run mean long-run variance frontier*

As before, with *incomplete* markets we face the same issue of finding the one of many possible discount factors $m$ which leads to a tradeable payoff. Again, however, we can use the quadratic utility approximation

$$\begin{aligned} u(c) &= -\frac{1}{2}\left(c^b - c\right)^2 \\ U &= \mathcal{E}\left[-\frac{1}{2}\left(c^b - c\right)^2\right] = E\sum_t \beta^t \left(-\frac{1}{2}\right)(c_t^b - c_t)^2 \end{aligned}$$

and the above analysis goes through *exactly.* Again, all we have to do is to reinterpret the symbols.

The optimal portfolio with a nonstochastic bliss point and no labor income is

$$\hat{y} = y^f + \frac{1}{\gamma}\left(y^f - y^*\right).$$

We recognize a long-run mean/long-run variance efficient portfolio on the right hand side. Aggreagting across identical individuals we have

$$\hat{y}^i = y^f + \frac{\gamma^a}{\gamma^i}\left(\hat{y}^m - y^f\right).$$

Thus, the classic propositions have straightforward reinterpretations:

1. *Each investor holds a portfolio on the long-run mean/ long-run variance frontier.*
2. *The market portfolio is also on the long-run mean / long-run variance frontier.*
3. *Each investor's portfolio can be spanned by a real perpetuity $y^f$ and a claim to aggregate consumption $\hat{y}^m$*

In addition a "long-run" version of the CAPM holds in this economy, since the market is "long-run" efficient.

Keep in mind that *all* of this applies with arbitrary return dynamics – we are *not* assuming iid returns – and it holds with incomplete markets, in particular that innovations to state variables are not traded. As conventional mean-variance theory gave a useful approximate *characterization* of optimal portfolios without actually *calculating* them – finding the mean-variance frontier is hard – so here we give an approximate *characterization* of optimal portfolios in a fully dynamic, intertemporal, incomplete markets context. Calculating them – finding $x^*$, $y^*$, the long run mean-long run variance frontier, or supporting a payoff $\hat{x}$ with dynamic trading in specific assets – will also be hard.

# 4 Portfolio theory by choosing portfolio weights

The standard approach to portfolio problems is quite different. Rather than summarize assets by a discount factor and choose the final *payoff*, you specify the assets explicitly and choose the portfolio weights.

## 4.1 One period, power-lognormal

---

I re-solve the one period, power utility, lognormal example by explicitly choosing portfolio weights. The answer is the same, but we learn how to implement the answer by dynamically trading the stock and bond. The portfolio holds a constantly-rebalanced share $\alpha_t = \frac{1}{\gamma}\frac{\mu-r}{\sigma^2}$ in the risky asset.

This is a classic theorem: the fraction invested in the risky asset is independent of investment horizon. It challenges conventional wisdom that young people should hold more stocks since they can afford to wait out any market declines.

---

This approach is easiest to illustrate in a canonical example, the power-lognormal case we have already studies. At each point in time, the investor puts a fraction $\alpha_t$ of his wealth in the risky asset. Thus the problem is

$$\max_{\{\alpha_t\}} Eu(c_T). \text{ s.t.}$$

$$\begin{aligned} dW_t &= W_t\left[\alpha_t \frac{dS_t}{S_t} + (1-\alpha_t) r dt\right] \\ c_T &= W_T;\ W_0 \text{ given} \end{aligned}$$

I start with the canonical lognormal iid environment,

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dz_t \\ \frac{dB}{B} &= r dt. \end{aligned}$$

Substituting, wealth evolves as

$$\frac{dW_t}{W_t} = \left[r + \alpha_t(\mu - r)\right] dt + \alpha_t \sigma dz. \tag{44}$$

We find the optimal weights $\alpha_t$ by dynamic programming. The value function satisfies

$$V(W,t) = \max_{\{\alpha_t\}} E_t V(W_{t+dt}, t+dt)$$

and hence, using Ito's lemma,

$$
\begin{aligned}
0 &= \max_{\{\alpha_t\}} E_t \left\{ V_W dW + \frac{1}{2} V_{WW} dW^2 + V_t dt \right\} \\
0 &= \max_{\{\alpha_t\}} W V_W \left[ r + \alpha_t (\mu - r) \right] + \frac{1}{2} W^2 V_{WW} \alpha_t^2 \sigma^2 + V_t \qquad (45)
\end{aligned}
$$

The first order condition for portfolio choice $\alpha_t$ leads directly to

$$
\alpha_t = -\frac{V_W}{W V_{WW}} \frac{\mu - r}{\sigma^2} \qquad (46)
$$

We will end up proving

$$
V(W, t) = k(t) W_t^{1-\gamma}
$$

and thus

$$
\alpha_t = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2}. \qquad (47)
$$

*The proportion invested in the risky asset is a constant, independent of wealth and investment horizon.* It is larger, the higher the stock excess return, lower variance, and lower risk aversion[2]. Conventional wisdom says you should invest more in stocks if you have a longer horizon; the young should invest in stocks, while the old should invest in bonds. The data paint an interesting converse puzzle: many young people invest in bonds until they build up a safe "nest egg," and the bulk of stock investment is done by people in their mid 50s and later. In this model, the conventional wisdom is wrong.

Of course, models are built on assumptions. A lot of modern portfolio theory is devoted to changing the assumptions so that the conventional wisdom is right, or so that the "safety-first" stylized fact is optimal. For example, time-varying expected returns can raise the Sharpe ratio of long-horizon investments, and so can make it optimal to hold more in stocks for longer investment horizons.

With the optimal portfolio weights in hand, invested wealth $W$ follows

$$
W_T = W_0 e^{(1-\alpha)\left(r + \frac{1}{2}\sigma^2 \alpha\right) T} \left( \frac{S}{S_0} \right)^{\alpha} \qquad (48)
$$

This is exactly the result we derived above. If $\alpha = 1$, we obtain $W = W_0 \left( S_T / S_0 \right)$, and if $a = 0$ we obtain $W_T = W_0 e^{rT}$, sensibly enough.

[2] Actually, the quantity $-\frac{V_W}{W V_{WW}}$ *is* the risk aversion coefficient. Risk aversion is often measured by people's resistance to taking bets. Bets affect your wealth, not your consumption, so aversion to wealth bets measures this quantity. The special result is that in this model, risk aversion is also equal to the local curvature of the utility function $\gamma$.

*Algebra* The algebra for (48) is straightforward if uninspiring.

$$\begin{aligned}
\frac{dW_t}{W_t} &= (1-\alpha)\,rdt + \alpha\frac{dS}{S} \\
d\ln W_t &= \frac{dW_t}{W_t} - \frac{1}{2}\frac{dW^2}{W^2} = (1-\alpha)\,rdt + \alpha\frac{dS_t}{S_t} - \frac{1}{2}\alpha^2\sigma^2 dt \\
d\ln S_t &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{dS^2}{S^2} = \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2 dt \\
d\ln W_t &= (1-\alpha)\,rdt + \alpha\left(d\ln S_t + \frac{1}{2}\sigma^2 dt\right) - \frac{1}{2}\alpha^2\sigma^2 dt \\
d\ln W_t &= \left[(1-\alpha)\,r + \frac{1}{2}\sigma^2\alpha(1-\alpha)\right] dt + \alpha d\ln S_t \\
d\ln W_t &= (1-\alpha)\left(r + \frac{1}{2}\sigma^2\alpha\right) dt + \alpha d\ln S_t \\
\ln W_T - \ln W_0 &= (1-\alpha)\left(r + \frac{1}{2}\sigma^2\alpha\right) T + \alpha\left(\ln S_T - \ln S_0\right) \\
W_T &= W_0 e^{(1-\alpha)\left(r+\frac{1}{2}\sigma^2\alpha\right)T}\left(\frac{S_T}{S_0}\right)^{\alpha}
\end{aligned}$$

*The value function* It remains to prove that the value function $V$ really does have the assumed form. This takes more algebra than intuition. Substituting the optimal portfolio $\alpha_t$ into( 45), The value function then solves the differential equation

$$\begin{aligned}
0 &= WV_W\left[r + \alpha_t(\mu - r)\right] + \frac{1}{2}W^2 V_{WW}\alpha^2\sigma^2 + V_t \\
0 &= \left[r - \frac{V_W}{WV_{WW}}\frac{(\mu-r)}{\sigma^2}(\mu-r)\right] + \frac{1}{2}\frac{W^2 V_{WW}}{WV_W}\left(\frac{V_W}{WV_{WW}}\frac{(\mu-r)}{\sigma^2}\right)^2\sigma^2 + \frac{V_t}{WV_W} \\
0 &= r - \frac{V_W}{WV_{WW}}\frac{(\mu-r)}{\sigma^2}(\mu-r) + \frac{1}{2}\frac{V_W}{WV_{WW}}\frac{(\mu-r)}{\sigma^2}^2 + \frac{V_t}{WV_W}
\end{aligned}$$

$$0 = r - \frac{1}{2}\frac{V_W}{WV_{WW}}\frac{(\mu-r)^2}{\sigma^2} + \frac{V_t}{WV_W}, \tag{49}$$

subject to the terminal condition

$$u(W_T) = V(W_T).$$

We guess a solution of the form

$$V(W,t) = e^{\eta(T-t)}W^{1-\gamma}$$

Hence,

$$\begin{aligned}
V_t &= -\eta e^{\eta(T-t)} W^{1-\gamma} \\
V_W &= (1-\gamma) e^{\eta(T-t)} W^{-\gamma} \\
V_{WW} &= -\gamma(1-\gamma) e^{\eta(T-t)} W^{-\gamma-1} \\
-\frac{V_W}{W V_{WW}} &= \frac{1}{\gamma} \\
\frac{V_t}{W V_W} &= -\frac{\eta}{1-\gamma}
\end{aligned}$$

Plugging in to the PDE (49), that equation holds if the undetermined coefficient $\eta$ solves

$$0 = r + \frac{1}{2}\frac{1}{\gamma}\frac{(\mu-r)^2}{\sigma^2} - \frac{\eta}{1-\gamma}$$

Hence,

$$\eta = (1-\gamma)\left[r + \frac{1}{2}\frac{1}{\gamma}\frac{(\mu-r)^2}{\sigma^2}\right]$$

and

$$V(W,t) = e^{(1-\gamma)\left[r+\frac{1}{2}\frac{1}{\gamma}\frac{(\mu-r)^2}{\sigma^2}\right](T-t)} W^{1-\gamma}$$

Since our guess works, the portfolio weights are in fact as given by equation (47). You might have guessed just $W^{1-\gamma}$, but having more time to trade and asset to grow makes success more likely.

## 4.2 Comparison with the payoff approach

Having both the discount factor approach and the portfolio weight approach in hand, you can see the appeal of the discount factor-complete markets approach. It took us two lines to get to $\hat{x} = (\text{const}) \times R_T^{\alpha}$, and only a few more lines to evaluate the constant in terms of initial wealth. The portfolio weight approach, by contrast took a lot of algebra. One reason it did so, is that we solved for a lot of stuff we didn't really need. We found not only the optimal *payoff*, but we found a specific dynamic trading strategy to support that payoff. That might be useful. On the other hand, you might want to implement the optimal payoff with a portfolio of call and put options at time zero and not have to spend the entire time dynamically trading. Or you might want to use 2 or 3 call options and then limit your amount of dynamic trading. The advantage of the portfolio choice approach is that you really know the answer is in the payoff space. The disadvantage is that if you make a slight change in the payoff space, you have to start the problem all over again.

Sometimes problems cannot be easily solved by choosing portfolio weights, yet we can easily characterize the payoffs. The habit example with $u'(c) = (c-h)^{-\gamma}$ above is one such example. We solved very quickly for final payoffs. You can try to solve this problem by choosing portfolio weights, but you will fail, in a revealing manner. Equation (**??**) will still describe portfolio weights. We had not used the form of the objective function in getting to this point. Now, however, the risk aversion coefficient will depend on wealth and time. If you are near $W = h$, you become much more risk averse! We need to solve the value function to see how much so. The differential equation for the value function (49) is also unchanged. The only thing that changes is the terminal condition. Now, we have a terminal condition

$$V(W,T) = (W-h)^{1-\gamma}.$$

Of course, our original guess $V(W,t) = e^{\eta(T-t)}W^{1-\xi}$ won't match this terminal condition. A natural guess $V(W,t) = e^{\eta(T-t)}(W - f(t)h)^{1-\gamma}$, alas, does not solve the differential equation. The only way I know to proceed analytically is to use the general solution of the differential equation

$$V(W,t) = \int a(\xi) e^{(1-\xi)\left[r+\frac{1}{2}\frac{1}{\gamma}\frac{(\mu-r)^2}{\sigma^2}\right](T-t)} W_t^{1-\xi} d\xi$$

and then find $a(\xi)$ to match the terminal condition. Not fun.

You can see the trouble. We have complicated the problem by asking not just for the answer – the time $T$ payoff or the number of contingent claims to buy –but also by asking for a trading strategy to synthesize those contingent claims from stock and bond trading. We achieved success by being able to stop and declare victory before the hard part. Certainly in this complete market model, it is simpler *first* to characterize the optimal payoff $\hat{x}$, and *then* to choose how to implement that payoff by a specific choice of assets, i.e. put and call options, dynamic trading, pure contingent claims, digital options, etc.

On the other hand, in general *incomplete* markets problems, choosing portfolio weights means you know you always stay in the asset space $\hat{x} \in \underline{X}$.

# 5 Dynamic intertemporal problems

Now we remove the iid assumption and allow mean returns, variance of returns and outside income to vary over time.

*Can we mix the one period and infinite period power lognormal?*

## 5.1 A single-variable Merton problem

---

We allow mean returns, return volatility and labor income to vary over time. This section simplifies by treating a single risky return and a single state variable. The optimal portfolio weight on the risky asset becomes

$$\alpha_t = \frac{1}{\gamma_t}\frac{\mu_t - r_t}{\sigma_t^2} + \eta_t \beta_{dy,dR}$$

where $\gamma_t$ and $\eta_t$ are risk aversion and aversion to the risk that the state variable changes, defined by corresponding derivatives of the value function, and $\beta_{dy,dR}$ is the regression coefficient of state-variable innovations on the risky return.

We see two new effects: 1) "Market timing." The allocation to the risky asset may rise and fall over time, for example if the mean excess return $\mu_t - r_t$ varies and $\gamma_t$ and $\sigma_t$ do not. 2) "Hedging" demand. If the return is good for "bad" realizations of the state variable, this raises the desirability and thus overall allocation to the risky asset.

These results simply characterize the optimal portfolio problem without solving for the actual value function. That step is much harder in general.

---

Here's the kind of portfolio problem we want to solve. We want utility over consumption, not terminal wealth; and we want to allow for time-varying expected returns and volatilities.

$$\max E \int_0^\infty e^{-\rho t} u(c_t) dt \text{ s.t.} \tag{50}$$

$$dR_t = \mu(y_t)dt + \sigma(y_t)dz_t \tag{51}$$

$$dy_t = \mu_y(y_t)dt + \sigma_y(y_t)dz_t \tag{52}$$

The objective can also be or include terminal wealth,

$$\max E \int_0^T e^{-\rho t} u(c_t) dt + EU(W_T).$$

In the traditional Merton setup, the $y$ variables are considered only as state variables for investment opportunities. However, we can easily extend the model to think of them as state variables for labor or proprietary income $e_t$ and include $c_t = x_t + e_t$ as well. I start in this section by specializing to a single state variable $y$, which simplifies the algebra and gives one set of classic results. The next section uses a vector of state variables and generates a different set of classic results.

If the investor puts weights $\alpha$ in the risky asset, wealth evolves as

$$\begin{aligned} dW &= W\alpha dR + W(1-\alpha)r dt + (e-c)\, dt \\ dW &= [Wr + W\alpha(\mu - r) + (e-c)]\, dt + W\alpha\sigma dz \end{aligned}$$

$e$ (really $e(y_t)$) is outside income.

The value function must include the state variable $y$, so the Bellman equation is

$$V(W,y,t) = \max_{\{c,\alpha\}} u(c)dt + E_t\left[e^{-\rho dt} V(W_{t+dt}, y_{t+dt}, t+dt)\right],$$

using Ito's lemma as usual,

$$\begin{aligned} 0 &= \max_{\{c,\alpha\}} u(c)dt - \rho V dt + V_t dt + V_W E_t(dW) + V_y E_t(dy) \\ &\quad + \frac{1}{2} V_{WW} dW^2 + \frac{1}{2} V_{yy} dy^2 + V_{Wy} dW dy. \end{aligned}$$

Next we substitute for $dW$, $dy$. The result is

$$0 = \max_{\{c,\alpha\}} u(c) - \rho V(W,y,t) + V_t + V_W\left[Wr + W\alpha\left(\mu - r\right) + e - c\right] + V_y\mu_y \qquad (53)$$
$$+\frac{1}{2}V_{WW}W^2\alpha^2\sigma^2 + \frac{1}{2}V_{yy}\sigma_y^2 + WV_{Wy}\alpha\sigma\sigma_y.$$

Now, the first order conditions. Differentiating (53),

$$\frac{\partial}{\partial c} : u'(c) = V_W$$

Marginal utility of consumption equals marginal value of wealth. A penny saved has the same value as a penny consumed.

Next, we find the first order condition for portfolio choice:

$$\frac{\partial}{\partial \alpha} \;:\; WV_W(\mu - r) + W^2V_{WW}\sigma^2\alpha + W\sigma\sigma_yV_{Wy} = 0$$
$$\alpha = -\frac{V_W}{WV_{WW}}\frac{(\mu - r)}{\sigma^2} - \frac{\sigma_y}{\sigma}\frac{V_{Wy}}{WV_{WW}}$$

This is the all-important answer we are looking for: the weights of the optimal portfolio. $\sigma\sigma_y = cov(dR, dy)$ is the covariance of return innovations with state variable innovations, so $\sigma\sigma_y/\sigma^2 = \beta_{dy,dR}$ is the regression coefficient of state variable innovations on return innovations. Thus, we can write the optimal portfolio weight in the risky asset as

$$\alpha = \left(-\frac{V_W}{WV_{WW}}\right)\frac{\mu_t - r_t}{\sigma_t^2} - \left(\frac{V_{Wy}}{WV_{WW}}\right)\beta_{dy,dR} \qquad (54)$$
$$= \frac{1}{\gamma}\frac{\mu_t - r_t}{\sigma_t^2} + \eta\beta_{dy,dR} \qquad (55)$$

In the second line, I have introduced the notation $\gamma$ for risk aversion and $\eta$ for "aversion" to state variable risk. The $\gamma$ here is not necessarily the power of a utility function; it is the local curvature of the value function at time $t$.

The first term is the same as we had before. However, the mean and variance change over time – that's the point of the Merton model. Thus, *Investors will "time the market," investing more in times of high mean or low variance.* The second term is new: *Investors will increase their holding of the risky asset if it covaries negatively with state variables of concern to the investor.* "Of concern" is measured by $V_{Wy}$. This is the "hedging" motive. A long term bond is a classic example. Bond prices go up when subsequent yields go down. Thus a long-term bond is an excellent hedge for the risk that interest rates decline, meaning your investment opportunities decline. Investors thus hold more long term bonds than they otherwise would, which may account for low long-term bond returns. Since stocks now mean-revert too, we should expect important quantitative results from the Merton model: *mean-reversion in stock prices will make stocks even more attractive.*

(This last conclusion depends on risk aversion, i.e. whether substitution or wealth effects dominate. Imagine that news comes along that expected returns are much higher. This has two effects. First there is a "wealth effect." The investor will be able to afford a lot more consumption in the future. But there is also a "substitution effect." At higher expected returns, it pays the investor to consume *less* now, and then consume even more in the future, having profited by high returns. If risk aversion, equal to intertemporal substitution, is high, the investor will not pay attention to the latter incentive. Raising consumption in the future means consumption rises now, so $V_W = u'(c)$ declines now, i.e. $V_{Wy} < 0$. However, if risk aversion is very low, the substitution effect will dominate. The investor consumers less now, so as to invest more. This means $V_W = u'(c)$ rises, and $V_{Wy} > 0$. Log utility is the knife edge case in which substitution and wealth effects offset, so $V_{Wy} = 0$. We usually think risk aversion is greater than log, so that case applies.)

Of course, risk aversion and state variable aversion are not constants, nor are they determined by preferences alone. This discussion presumes that risk aversion and state variable aversion do not change. They may. Only by fully solving the Merton model can we really see the portfolio implications.

*Completing the Merton model.*

Conceptually this step is simple, as before: we just need to find the value function. We plug optimal portfolio and consumption decisions into 53 and solve the resulting partial differential equation. However, even a brief look at the problem will show you why so little has been done on this crucial step, and thus why *quantitative* use of Merton portfolio theory languished for 20 years until the recent revival of interest in approximate solutions. The partial differential equation is, from (53) (algebra below)

$$\begin{aligned} 0 &= u(u'^{-1}(V_W)) - \rho V + V_t + WV_W r - V_W u'^{-1}(V_W) + V_W e + V_y \mu_y + \frac{1}{2} V_{yy} \sigma_y^2 \\ &\quad - \frac{1}{2} \frac{1}{\sigma^2 V_{WW}} \left[ V_W(\mu - r) + \sigma \sigma_y V_{Wy} \right]^2 . \end{aligned}$$

This is not a pleasant partial differential equation to solve, and analytic solutions are usually not available. The nonlinear terms $u(u'^{-1}(V_W))$ and $u'^{-1}(V_W)$ are especially troublesome, which accounts for the popularity of formulations involving the utility of terminal wealth, for which these terms are absent.

There are analytical solutions for the following special cases:

1. Power utility, infinite horizon, no state variables. As you might imagine, $V(W) = W^{1-\gamma}$ works again. This is a historically important result as it establishes that the CAPM holds even with infinitely lived, power utility investors, so long as returns are i.i.d. over time and there is no labor income. I solve it in the next subsection

2. Log utility, no labor income. In this case, $V_{Wy} = 0$, the investor does no intertemporal hedging. Now we recover the log utility CAPM, even when there are state variables.

3. Power utility of terminal wealth (no consumption), $AR(1)$ state variable, no labor income, (Kim and Omberg 1996). Here the natural guess that $V(W, y, t) = f(t)V(W)(a + by + cy^2)$ works.

*The Algebra:* Plugging optimal consumption $c$ and portfolio $\alpha$ decisions into (53),

$$\begin{aligned} 0 &= u(c) - \rho V + V_t + V_W \left[ Wr + W\alpha(\mu - r) + e - c \right] + V_y \mu_y \\ &+ \frac{1}{2} V_{WW} W^2 \alpha^2 \sigma^2 + \frac{1}{2} V_{yy} \sigma_y^2 + W V_{Wy} \alpha \sigma \sigma_y \end{aligned}$$

$$\begin{aligned} 0 &= u(u'^{-1}(V_W)) - \rho V + V_t + W V_W r - V_W u'^{-1}(V_W) + V_W e + V_y \mu_y + \frac{1}{2} V_{yy} \sigma_y^2 \\ &+ \frac{1}{2} V_{WW} W^2 \alpha^2 \sigma^2 + W \left( V_W(\mu - r) + V_{Wy} \sigma \sigma_y \right) \alpha \end{aligned}$$

$$\begin{aligned} 0 &= u(u'^{-1}(V_W)) - \rho V + V_t + W V_W r - V_W u'^{-1}(V_W) + V_W e + V_y \mu_y + \frac{1}{2} V_{yy} \sigma_y^2 \\ &+ \frac{1}{2} V_{WW} W^2 \sigma^2 \left[ \frac{V_W}{W V_{WW}} \frac{(\mu - r)}{\sigma^2} + \frac{\sigma \sigma_y}{\sigma^2} \frac{V_{Wy}}{W V_{WW}} \right]^2 \\ &- W \left[ V_W(\mu - r) + V_{Wy} \sigma \sigma_y \right] \left[ \frac{V_W}{W V_{WW}} \frac{(\mu - r)}{\sigma^2} + \frac{\sigma \sigma_y}{\sigma^2} \frac{V_{Wy}}{W V_{WW}} \right] \end{aligned}$$

$$\begin{aligned} 0 &= u(u'^{-1}(V_W)) - \rho V + V_t + W V_W r - V_W u'^{-1}(V_W) + V_W e + V_y \mu_y + \frac{1}{2} V_{yy} \sigma_y^2 \\ &+ \frac{1}{2} \frac{1}{\sigma^2 V_{WW}} \left[ V_W(\mu - r) + \sigma \sigma_y V_{Wy} \right]^2 \\ &- \frac{1}{\sigma^2 V_{WW}} \left[ V_W(\mu - r) + V_{Wy} \sigma \sigma_y \right] \left[ V_W(\mu - r) + \sigma \sigma_y V_{Wy} \right] \end{aligned}$$

$$\begin{aligned} 0 &= u(u'^{-1}(V_W)) - \rho V + V_t + W V_W r - V_W u'^{-1}(V_W) + V_W e + V_y \mu_y + \frac{1}{2} V_{yy} \sigma_y^2 \\ &- \frac{1}{2} \frac{1}{\sigma^2 V_{WW}} \left[ V_W(\mu - r) + \sigma \sigma_y V_{Wy} \right]^2 . \end{aligned}$$

## 5.2 The power-lognormal iid model with consumption

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I solve the power utility infinite-horizon model with iid returns and no outside income. The investor consumes a constant proportion of wealth, and invests a constant share in the risky asset.

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In the special case of power utility, no outside income and iid returns, the differential equation (53) specializes to

$$0 = \frac{V_W^{-\frac{1}{\gamma}(1-\gamma)}}{1-\gamma} - \rho V + V_t + W V_W r - V_W V_W^{-\frac{1}{\gamma}} - \frac{1}{2} \frac{1}{\sigma^2 V_{WW}} \left[ V_W(\mu - r) \right]^2$$

To solve it, we guess a functional form

$$V = k\frac{W^{1-\gamma}}{1-\gamma}.$$

Plugging in, we find that the differential equation holds if

$$k^{-\frac{1}{\gamma}} = \frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left[r + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right].$$

Hence, we can fully evaluate the policy: Optimal consumption follows

$$c = V_W^{-\frac{1}{\gamma}} = \frac{1}{\gamma}\left[\rho - (1-\gamma)\left(r + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right)\right]W \tag{56}$$

*The investor consumes a constant share of wealth $W$.* For log utility ($\gamma = 1$) we have $c = \rho W$. The second term only holds for utility different than log. If $\gamma > 1$, higher returns (either a higher risk free rate or the higher squared Sharpe ratio in the second term) lead you to raise consumption. Income effects are greater than substitution effects (high $\gamma$ resists substitution), so the higher "wealth effect" means more consumption now. If $\gamma < 1$, the opposite is true; the investor takes advantage of higher returns by consuming less now, building wealth up faster and then consuming more later. The risky asset share is, from (54),

$$\alpha = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}. \tag{57}$$

We already had the optimal consumption stream in (40). What we learn here is that we can support that stream by the consumption rule (56) and portfolio rule (57).

*The Algebra*

$$\begin{aligned} V &= k\frac{W^{1-\gamma}}{1-\gamma} \\ V_W &= kW^{-\gamma} \\ V_{WW} &= -\gamma k W^{-\gamma-1} \end{aligned}$$

$$\begin{aligned} 0 &= \frac{k^{-\frac{1}{\gamma}(1-\gamma)}W^{(1-\gamma)}}{1-\gamma} - \rho k\frac{W^{1-\gamma}}{1-\gamma} + WkW^{-\gamma}r - \left(kW^{-\gamma}\right)^{1-\frac{1}{\gamma}} + \frac{1}{2}\frac{\left(kW^{-\gamma}\right)^2}{\gamma k W^{-\gamma-1}}\frac{(\mu-r)^2}{\sigma^2} \\ 0 &= \frac{k^{1-\frac{1}{\gamma}}}{1-\gamma}W^{1-\gamma} - \frac{\rho k}{1-\gamma}W^{1-\gamma} + rkW^{1-\gamma} - k^{1-\frac{1}{\gamma}}W^{1-\gamma} + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{k}{\gamma}W^{1-\gamma} \\ 0 &= \frac{k^{-\frac{1}{\gamma}}}{1-\gamma} - \frac{\rho}{1-\gamma} + r - k^{-\frac{1}{\gamma}} + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2} \\ 0 &= \left(\frac{\gamma}{1-\gamma}\right)k^{-\frac{1}{\gamma}} - \frac{\rho}{1-\gamma} + r + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2} \\ k^{-\frac{1}{\gamma}} &= \frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left[r + \frac{1}{2}\frac{(\mu-r)^2}{\gamma\sigma^2}\right] \end{aligned}$$

## 5.3 Multivariate Merton problems and the ICAPM

We characterize the infinite-period portfolio problem with multiple assets and multiple state variables. The (conditional) multifactor efficient frontier emerges.

Now, let's solve the same problem with a vector of asset returns and a vector of state variables. This generalization allows us to think about how the investor's choice *among* assets may be affected by time-varying investment opportunities and by labor income and state variables for labor income. We start as before,

$$\max E \int_0^\infty e^{-\rho t} u(c_t) \text{ s.t.} \tag{58}$$

$$dR_t = \mu(y_t)dt + \sigma(y_t)dz_t \tag{59}$$

$$dy_t = \mu_y(y_t)dt + \sigma_y(y_t)dz_t \tag{60}$$

$$de_t = \mu_e(y_t)dt + \sigma_e(y_t)dz_t \tag{61}$$

Now I use $dR$ to denote the vector of $N$ returns $dS_i/S_i$, so $\mu$ is an $N$ dimensional vector. $y$ is a $K$ dimensional vector of state variables. $dz$ is an (at least) $N+K$ dimensional vector of independent shocks, $E_t(dzdz') = I$. Thus, $\sigma$ is an $N \times (N+K)$ dimensional matrix and $\sigma_y$ is a $K \times (N+K)$ dimensional matrix. I'll examine the case in which one asset is a risk free rate, $r_t$ Since it varies over time, it is one of the elements of $y_t$. The conventional statement of the problem ignores outside income and only thinks of state variables that drive the investment opportunity set, but since labor income is important and all the results we will get to accommodate it easily, why not include it.

Now, if the investor puts weights $\alpha$ on the risky assets, wealth evolves as

$$\begin{aligned} dW &= W(\alpha' dR) + W(1 - 1'\alpha)rdt + (e-c)\,dt \\ dW &= [Wr + W\alpha'(\mu - r) + (e-c)]\,dt + W\alpha'\sigma dz. \end{aligned}$$

The Bellman equation is

$$V(W,y,t) = \max_{\{c,\alpha\}} u(c)dt + E_t\left[e^{-\rho dt}V(W_{t+dt}, y_{t+dt}, t+dt)\right],$$

and using Ito's lemma as usual,

$$\begin{aligned} 0 &= \max_{\{c,\alpha\}} u(c)dt - \rho V dt + V_t dt + V_W E_t(dW) + V_{y'}E_t(dy) \\ &\quad + \frac{1}{2}V_{WW}dW^2 + \frac{1}{2}dy'V_{yy'}dy + dWV_{Wy'}dy. \end{aligned}$$

I use the notation $V_{y'}$ to denote the row vector of derivatives of $V$ with respect to $y$. $V_y$ would be a corresponding column vector. $V_{yy'}$ is a matrix of second partial derivatives.

Next we substitute for $dW$, $dy$. The result is

$$\begin{aligned} 0 &= \max_{\{c,\alpha\}} u(c) - \rho V(W,y,t) + V_t + V_W \left[ Wr + W\alpha' \left( \mu - r1 \right) - c \right] + V_{y'}\mu_y \\ &+ \frac{1}{2} V_{WW} W^2 \alpha' \sigma\sigma' \alpha + \frac{1}{2} Tr(\sigma_y' V_{yy'} \sigma_y) + W\alpha'\sigma\sigma_y' V_{Wy} \end{aligned} \tag{62}$$

This is easy except for the second derivative terms. To derive them

$$E(dz'Adz) = \sum_{i,j} dz_i A_{ij} dz_j = \sum_i A_{ii} = Tr(A).$$

Then,

$$dy'V_{yy'}dy = (\sigma_y dz)'V_{yy'}(\sigma_y dz) = dz'\sigma_y' V_{yy'} \sigma_y dz = Tr(\sigma_y' V_{yy'} \sigma_y).$$

We can do the other terms similarly,

$$\begin{aligned} dWV_{Wy'}dy &= (W\alpha'\sigma dz)'V_{Wy'}(\sigma_y dz) = Wdz'\sigma'\alpha V_{Wy'}\sigma_y dz \\ &= WTr(\sigma'\alpha V_{Wy'}\sigma_y) = WTr(\alpha'\sigma\sigma_y' V_{Wy}) = W\alpha'\sigma\sigma_y' V_{Wy} \\ V_{WW}dW^2 &= V_{WW}(W\alpha'\sigma dz)'(W\alpha'\sigma dz) = W^2 V_{WW} dz'\sigma'\alpha\alpha'\sigma dz \\ &= W^2 V_{WW} Tr\left(\sigma'\alpha\alpha'\sigma\right) = W^2 V_{WW} Tr\left(\alpha'\sigma\sigma'\alpha\right) = W^2 V_{WW} \alpha'\sigma\sigma'\alpha \end{aligned}$$

(I used $Tr(AA') = Tr(A'A)$ and $Tr(AB) = Tr(A'B')$. These facts about traces let me condense a $(N+K)\times(N+K)$ matrix to a $1\times 1$ quadratic form in the last line, and let me transform from an expression for which it would be hard to take $\alpha$ derivatives, $Tr(\sigma'\alpha\alpha'\sigma)$, to one that is easy, $\alpha'\sigma\sigma'\alpha$).

Now, the first order conditions. Differentiating (62), we obtain again

$$\frac{\partial}{\partial c} : u'(c) = V_W$$

Differentiating with respect to $\alpha$,

$$\begin{aligned} \frac{\partial}{\partial \alpha} &: \quad WV_W(\mu - r1) + W^2 V_{WW}\sigma\sigma'\alpha + W\sigma\sigma_y' V_{Wy} = 0 \\ \alpha &= -\frac{V_W}{WV_{WW}} \left(\sigma\sigma'\right)^{-1} (\mu - r) - \left(\sigma\sigma'\right)^{-1} \sigma\sigma_y' \frac{V_{Wy}}{WV_{WW}} \end{aligned}$$

This is the all-important answer we are looking for: the weights of the optimal portfolio. It remains to make it more intuitive. $\sigma\sigma' = cov(dR, dR') = \Sigma$ is the return innovation covariance matrix. $\sigma\sigma_y' = cov(dR, dy') = \sigma_{dR,y'}$ is the covariance of return innovations with state variable innovations, and $\left(\sigma\sigma'\right)^{-1}\sigma\sigma_y' = \Sigma^{-1}\sigma_{dR,y'} = \beta_{dy,dR}'$ is a matrix of multiple regression coefficients of state variable innovations on return innovations. Thus, we can write the optimal portfolio weights as

$$\alpha = -\frac{V_W}{WV_{WW}} \Sigma^{-1}(\mu - r1) - \beta_{dy,dR}' \frac{V_{Wy}}{WV_{WW}} \tag{63}$$

The first term is exactly the same as we had before, generalized to multiple assets. We recognize in $\Sigma^{-1}(\mu - r1)$ the weights of a mean-variance efficient portfolio. Thus we obtain an important result: *In an iid world, investors will hold an instantaneously mean-variance efficient portfolio.* Since we're using diffusion processes which are locally normal, this is the proof behind the statement that normal distributions result in mean-variance portfolios. Mean variance portfolios do not *require* quadratic utility, which I used above to start thinking about mean-variance efficiency. However, note that even if $\alpha$ is constant over time, this means dynamically trading and rebalancing, so that portfolios will not be mean-variance efficient at discrete horizons. In addition, the risky asset share $\alpha$ will generally change over time, giving even more interesting and mean-variance inefficient discrete-horizon returns.

The second term is new: *Investors will shift their portfolio weights towards assets that covary with, and hence can hedge, outside income or changes in the investment opportunity set.* Investors will differ in their degree of risk aversion and "aversion to state variable risk" so we can write the optimal portfolio as

$$\alpha = \frac{1}{\gamma}\Sigma^{-1}(\mu - r1) + \beta'_{dy,dR}\eta \tag{64}$$

where again

$$\gamma \equiv -\frac{WV_{WW}}{V_W}$$

is the investor's risk aversion, and

$$\eta = -\frac{V_{Wy}}{WV_{WW}}$$

is the investor's "aversion to state variable risk."

*Multifactor efficiency and $K+2$ funds.*

We can nicely interpret this result as a generalization of mean-variance portfolio theory, following Fama (1996). *The Merton investor minimizes the variance of return subject to mean return, and subject to the constraint that returns have specified covariance with innovations to state variables.* Let's form portfolios

$$dR^p = \alpha' dR + (1 - \alpha' 1) r dt$$

The suggested mean, variance, covariance problem is

$$\begin{aligned} \min var_t(dR^p) \text{ s.t. } E_t dR^p &= E; \quad cov_t(dR^p, dy) = \xi \\ \min_{\{\alpha\}} \alpha'\sigma\sigma'\alpha \text{ s.t.} r + \alpha'(\mu - r1) &= E \; ; \; \alpha'\sigma\sigma'_y = \xi \end{aligned}$$

Introducing Lagrange multipliers $\lambda_1, \lambda_2$, the first order conditions are

$$\begin{aligned} \sigma\sigma'\alpha &= \lambda_1(\mu - r1) + \sigma\sigma'_y\lambda_2 \\ \alpha &= \lambda_1(\sigma\sigma')^{-1}(\mu - r1) + (\sigma\sigma')^{-1}\sigma\sigma'_y\lambda_2 \end{aligned} \tag{65}$$

This is exactly the same answer as (63)!

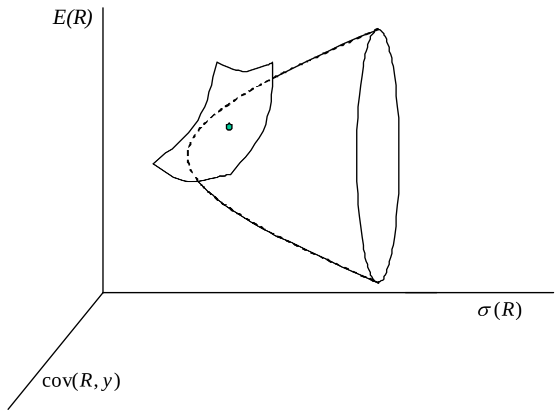


Figure 5: Multifactor effcient portfolio and "indifference curve."

Figure 5 illustrates. As the mean-variance frontier is a hyperbola, the mean-variance-covariance frontier is a revolution of a hyperbola. Fama calls this frontier the set of *multifactor efficient portfolios.* (Covariance with a state variable is a linear constraint on returns, as is the mean. Thus, the frontier is the revolution of a parabola in mean-variance-covariance space, and the revolution of hyperbola in mean-standard deviation-covariance space as shown. I draw the prettier case with no risk free rate. With a risk free rate, the frontier is a cone.) As shown in the picture, we can think of the investor as maximizing preferences defined over mean, variance and covariance of the portfolio, just as previously we could think of the investor as maximizing preferences defined over mean and variance of the portfolio.

The first term in (63) and (65) is the mean-variance frontier, or a tangency portfolio. (Set $\lambda_2 = 0$ and equation (65) derives this result.) Thus, we see that *typical investors do not hold mean-variance efficient portfolios.* They are willing to give up some mean-variance efficiency in return for a portfolio that hedges the state variable innovations $dy$.

What do they hold? Mean-variance portfolio theory led to the famous "two fund" theorem. This generalization leads naturally to a $K+2$ fund theorem. *Investors splits their wealth between the tangency portfolio and $K$ mimicking portfolios for state variable innovation risk.* To see this, let's write the investor's optimal portfolio return, not just its weights.

$$\begin{aligned} dR_i &= \alpha^{i\prime} dR + (1 - \alpha^{i\prime} 1) r dt; \\ &= r dt + \alpha^{i\prime} (dR - 1 r dt) \end{aligned}$$

In the latter expression, I split up the investor's portfolio into a risk free investment and an investment in a zero cost portfolio. Following (64), we can split up this portfolio return

$$\begin{aligned} dR_i &= rdt + \frac{1}{\gamma^i} dR^T + \eta^{i\prime} dR^H \\ dR^T &= (\mu - r1)'\Sigma^{-1}(dR - r1dt) \qquad (66) \\ dR^H &= \beta_{dy,dR}(dR - r1dt) \qquad (67) \end{aligned}$$

We recognize $dR^T$ as a zero-cost investment in a mean-variance efficient or tangency portfolio. The $dR^H$ portfolios are zero cost portfolios formed from the fitted values of regressions of state variable innovations on the set of asset returns. They are *mimicking portfolios* for the state variable innovations, projection of the state variable innovations on the payoff space. They are also "maximum correlation" portfolios, as regression coefficients minimize residual variance, $\min_{\{\beta_{dy,dR}\}} var\left(dy - \beta_{dy,dR} dR\right)$. Of course, any two K+2 independent multifactor-efficient portfolios will span the multifactor efficient frontier, so you may see other expressions. The key is to find an *interesting* set of portfolios that span the frontier.

*The ICAPM and the market portfolio.*

It's always interesting to express portfolio theory with reference to the market portfolio. The average investor must hold the market portfolio. The market portfolio is the average of individual portfolios, weighted by wealth $\alpha^m = \sum_i W^i \alpha^i / \sum_i W^i$. Thus, summing (64) over investors,

$$\alpha^m = \frac{1}{\gamma^m}\Sigma^{-1}(\mu - r1) + \Sigma^{-1}\sigma_{dR,dy'}\eta^m \qquad (68)$$

The ICAPM solves for the mean excess return $\mu - r1$,

$$\mu - r1 = \gamma^m \Sigma \alpha^m - \sigma_{dR,dy'}\eta^m\gamma^m$$

The market portfolio return is

$$dR^m = \alpha^{m\prime} dR + (1 - \alpha^{m\prime}1)rdt$$

Thus, we recognize

$$\Sigma\alpha^m = cov(dR, dR')\alpha^m = cov(dR, dR^m)$$

and we have *The ICAPM:*

$$\mu - r1 = \gamma^m cov(dR, dR^m) - cov(dR, dy')\gamma^m\eta^m$$

*mean excess returns are driven by covariance with the market portfolio and covariance with each of the state variables.* The risk aversion and state-variable aversion coefficients give the slopes of average return on covariances.

This expression with covariance on the right hand side is nice, since the slopes are related to preference (well, value function) parameters. However, it's traditional to express the right hand side in terms of regression betas, and to forget about the economic interpretation of the $\lambda$ slope coefficients. This is easy to do:

$$\begin{aligned}
\mu - r1 &= \beta_{dR,dR^m}\lambda_m - \beta_{dR,dy'}\lambda_{dy} \qquad (69) \\
\beta_{dR,dR^m} &= \frac{cov(dR, dR^m)}{\sigma^2_{dR^m}}; \\
\beta_{dR,dy'} &= (cov(dy, dy')^{-1} cov(dy, dR'))' \\
\lambda_m &= \frac{\sigma^2_{dR^m}}{\gamma^m}; \\
\lambda_{dy} &= cov(dy, dy')\frac{\eta^m}{\gamma^m}
\end{aligned}$$

Now we have expected returns as a linear function of market (total wealth) betas, *and* betas on state variable innovations (or their mimicking portfolios).

Don't forget that all the moments are conditional! The whole point of the ICAPM is that at least one of the conditional mean or conditional variance must vary through time.

*More Portfolio Implications*

Return to the market portfolio in (71) or (68). The first term – and only the first term – gives the mean-variance efficient portfolio. Thus, *the market portfolio is no longer mean-variance efficient.* Referring to Figure 5, you can see that the optimal portfolio has slid down from the vertical axis of the nose-cone shaped multifactor efficient frontier. The average investor, and hence the market portfolio, gives up some mean-variance efficiency in order to gain a portfolio that better hedges the state variables.

This prediction is the source of much portfolio advice from the ICAPM, for example, why the ICAPM interpretation of the Fama-French 3 factor model, is used as a sales tool for value-stock portfolios. If you find a mean-variance investor, an investor who does not fear the state variable changes and so has $\eta = 0$; this investors can now profit by deviating from market weights. He should slide up the nose-cone shaped multifactor efficient frontier in Figure **??**, in effect selling state-variable insurance to other investors, and charging a fee to do so.

Precisely, we can rewrite the optimal portfolio for a mean-variance investor as

$$dR^i = \frac{\gamma^m}{\gamma^i} dR^m - \frac{\gamma^m}{\gamma^i}\eta^{m\prime} dR^H \qquad (70)$$

where $dR^H$ are the state-variable mimicking or hedge portfolios from (67). This expression tells us, *quantitatively*, how the mean-variance investor should deviate from the market portfolio in order to profit from the ICAPM. As you can see, the investor should buy (or sell) some of the hedge portfolios in addition to the market portfolios. An estimate of the ICAPM will tell us the slope coefficients (of average returns on covariances) $\gamma^m, \eta^m$.

To get to (70), start with the optimal portfolio for a mean-variance investor,

$$dR^i_i = rdt + \frac{1}{\gamma^i}(\mu - r1)'\Sigma^{-1}(dR - r1dt)$$

This equation alone is not very inspiring – we have just written down the condition for mean-variance efficiency. The point of the ICAPM or any other model is to help us to identify a mean-variance efficient portfolio. The market portfolio return is

$$dR^m = rdt + \alpha^{m\prime}\left(dR - r1dt\right)$$

$$dR^m = rdt + \left[\frac{1}{\gamma^m}(\mu - r1)'\Sigma^{-1} + \eta^{m\prime}\beta_{dy,dR}\right]\left(dR - rdt\right) \tag{71}$$

We can solve (71) for the part we're looking for

$$(\mu - r)'\Sigma^{-1}\left(dR - rdt\right) = \gamma^m\left(dR^m - rdt\right) - \gamma^m\eta^{m\prime}\beta_{dy,dR}\left(dR - rdt\right) \tag{72}$$

Thus, the portfolio weights for our mean-variance efficient investor are

$$\begin{aligned} dR^i &= \frac{\gamma^m}{\gamma^i}dR^m - \frac{\gamma^m}{\gamma^i}\eta^{m\prime}\beta_{dy,dR}\left(dR - rdt\right) \\ dR^i &= \frac{\gamma^m}{\gamma^i}dR^m - \frac{\gamma^m}{\gamma^i}\eta^{m\prime}dR^H \end{aligned}$$

The trouble of course is that *everybody* can't be a mean-variance investor, or the ICAPM would not hold. This advice must hold for the measure zero set of truly mean variance investors, and among those only the ones who have not already optimized. *There are no portfolio implications of the ICAPM for the average investor.* The average investor must hold the market portfolio! If this advice is successful, we should question why there are so many mean-variance investors out there if the equilibrium is truly a multifactor equilibrium!

More generally, however, we express the individual's portfolio in terms of the market portfolio rather than the tangency portfolio. Expressing $i$s portfolio in terms of the market portfolio, and using (72)

$$\begin{aligned} dR^i &= rdt + \left(\frac{1}{\gamma^i}(\mu - r1)'\Sigma^{-1} + \eta^{i\prime}\beta_{dy,dR}\right)\left(dR - rdt\right) \\ dR^i &= rdt + \frac{\gamma^m}{\gamma^i}\left(dR^m - rdt\right) + \left(\eta^{i\prime} - \frac{\gamma^m}{\gamma^i}\eta^{m\prime}\right)dR^H \end{aligned}$$

The investor starts with the market portfolio and then buys or sells the state variable hedge

portfolios as his risk aversion and aversion to state variable risk differs from that of the market average.

This is a lovely expression. It emphasizes that as many investors should want to buy as to sell the hedge portfolios. Also, it emphasizes that *hedge portfolios do not have to be priced to be interesting.* A hedge portfolio that is not priced – for which $\eta^m = 0$ – still shows up in every investor's portfolio. Half are long, and half are short. It's not interesting for the last remaining *mean-variance* investor, but they are a vanishing breed. Getting the shorts to sell to the longs, and charging a fee along the way, is a much more interesting business than pursuing ephemeral state variable risk premia for mean variance investors. For example, idiosyncratic labor income risk is not priced. Yet there should be far more labor income hedging in asset markets than is currently practiced.

*Completing the Merton model.*

Again, we still need to compute the levels of risk aversion and "state variable aversion" from the primitives of the model, the utility function and formulas for the evolution of stock prices. Conceptually this step is simple, as before: we just need to find the value function. Alas, the resulting partial differential equation is so ugly that work on this multivariate model has pretty much stopped at the above qualitative analysis. From (53), the equation is

$$\begin{aligned} 0 &= u(u'^{-1}(V_W)) - \rho V + V_t + V_W W r - V_W u'^{-1}(V_W) + V_{y'}\mu_y + \frac{1}{2}Tr(\sigma_y' V_{yy'}\sigma_y) \\ &\quad + W\alpha^{*\prime}\left[(\mu - r1)V_W + \sigma\sigma_y' V_{Wy}\right] + \frac{1}{2}V_{WW}W^2\alpha^{*\prime}\sigma\sigma'\alpha^* \end{aligned}$$

where

$$\alpha^* = -\frac{V_W}{W V_{WW}}(\sigma\sigma')^{-1}(\mu - r1) - (\sigma\sigma')^{-1}\sigma\sigma_y'\frac{V_{Wy}}{W V_{WW}}$$