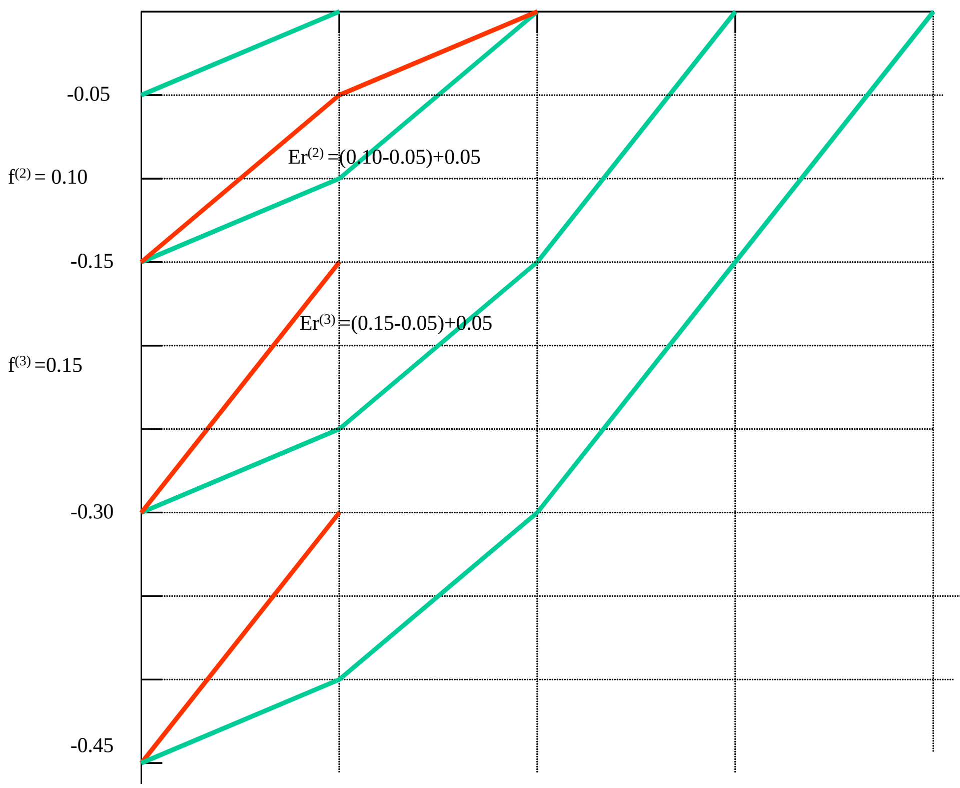
## Problem Set 10 answers

1. Answer: This includes a 4 year bond that I deleted from the question.

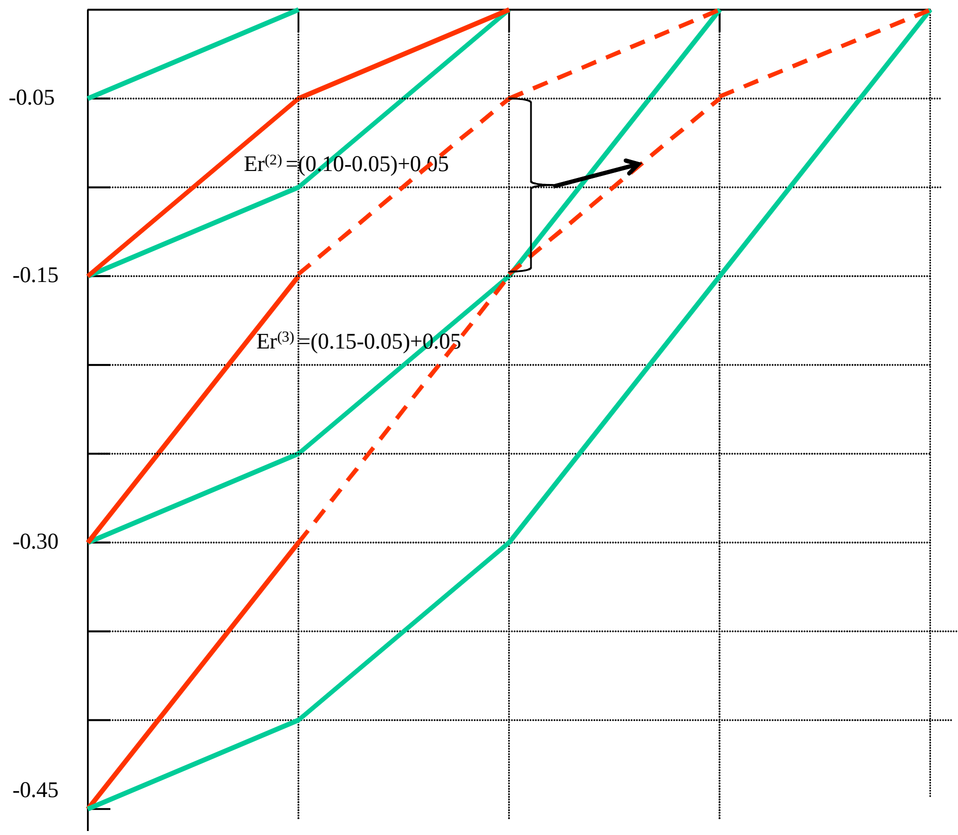
$$
\begin{aligned}
y^{(1)} &= 0.05 \\
y^{(2)} &= 0.075 \\
y^{(3)} &= 0.10
\end{aligned}
$$

$$
\begin{aligned}
f^{(2)} &= 0.10 \\
f^{(3)} &= 0.15
\end{aligned}
$$

I drew expectations just by having each line rise at the same rate as the one year rate that year. (green) for FB, $rx_{t+1}^{(n)} = 0 + 1(f_t^{(n)} - y_t^{(1)})$ as plotted




Now, why didn't I ask you to plot the bonds through time completely? A fun extra problem – the one-year Fama Bliss coefficients cannot be all 1.0. Here's what happens if you continue this way – in each period, use that period's forward rate to forecast the return on that bond, as shown by the } and arrow. Notice that the one-year rate *never* changes. Well, that's what we said, if it's always 1.0 and 0.0, then the one year rate never rises. (The top right dashed red line is still growing at 5%.) But this contradicts the longer-term FB finding that by 5 years, the one year rate *does* rise. So, it really must be 0.8 and 0.2 (or so) in the one-year forecasts, so those 0.2 can build up to the long-term rise in short rates.




(a) Under expectations, forward rates obey

$$f_t^{(n)} = E_t\left(f_{t+1}^{(n-1)}\right) = E_t\left(f_{t+2}^{(n-2)}\right) = E_t\left(y_{t+n-1}^{(1)}\right)$$

Proof.

$$\begin{aligned} f_t^{(n)} &= E_t y_{t+n-1}^{(1)} \\ f_{t+1}^{(n-1)} &= E_{t+1} y_{t+n-1}^{(1)} \end{aligned}$$

Take $E_t$ of both sides of the second equation.
Now, expessing it in the way of the question,

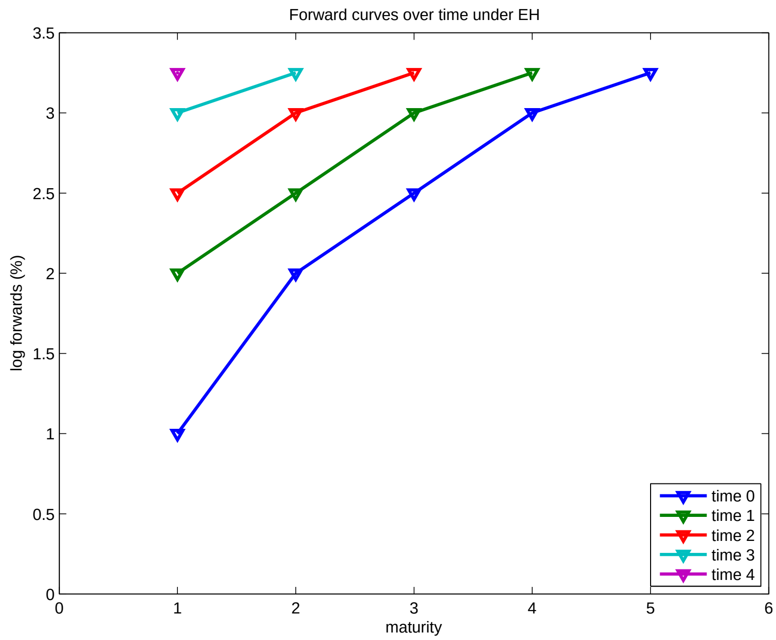
$$f_t^{(n+k)} = E_t\left(f_{t+k}^{(n)}\right)$$

or.

$$f_t^{(n)} = E_t\left(f_{t+k}^{(n-k)}\right).$$

In English: The rate at which you can contract *today* to borrow from time $t+n-1$ and repay at time $t+n$ must equal the expected rate at which you can contract *tomorrow* to borrow over the same period, and both are equal to the expected spot rate when the time comes. Well, now that you put it that way, duh.

(b) Here is the numerical example, with initial forward rates (1%, 2%, 2.5%,3%, 3.25%) You can see that the forward curve is expected to *shift to the left,* which means that if it is rising, that the yield curve at every maturity is expected to *go up.* (We don't know what will happen to maturities over to the right of those indicated.)




(c) Differencing, under expectations,

$$E_t\left(f_{t+1}^{(n)}\right) - f_t^{(n)} = f_t^{(n+1)} - f_t^{(n)}$$

Thus, *if the forward curve is upward sloping we expect all forward rates to rise.*

(d) If Fama and Bliss are right, we have instead

$$E_t\left(f_{t+1}^{(n-1)}\right) = f_t^{(n-1)}$$

Proof:

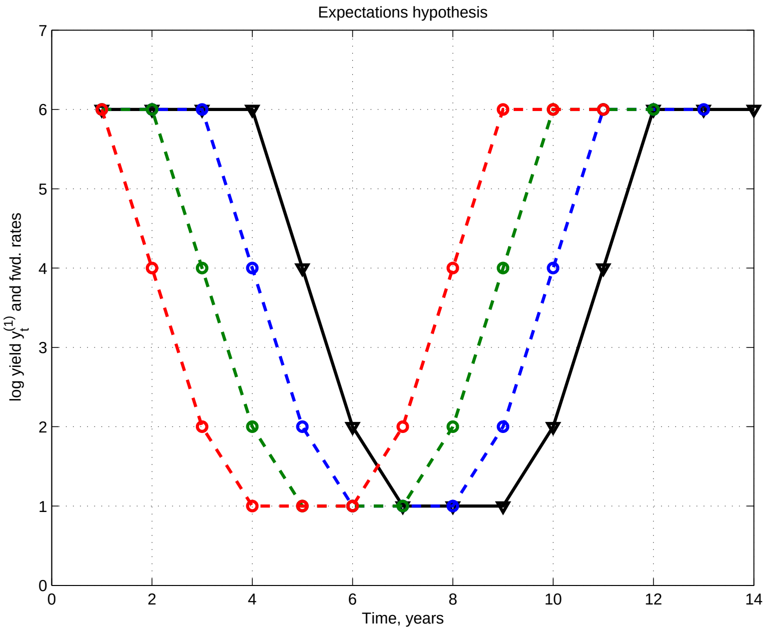
$$\begin{aligned} Er_{t+1}^{(n)} - y_t^{(1)} &= f_t^{(n)} - y_t^{(1)} \\ E_t p_{t+1}^{(n-1)} - p_t^{(n)} - y_t^{(1)} &= f_t^{(n)} - y_t^{(1)} \\ E_t p_{t+1}^{(n-2)} - p_t^{(n-1)} - y_t^{(1)} &= f_t^{(n-1)} - y_t^{(1)} \\ E_t f_{t+1}^{(n-1)} - f_t^{(n)} &= f_t^{(n-1)} - f_t^{(n)} \\ E_t f_{t+1}^{(n-1)} &= f_t^{(n-1)} \end{aligned}$$

This nicely generalizes the FB view – everything is a random walk, not just the one year rate. Now, *all forward rates are expected to stay the same, no matter what the yield curve does*
Under FB, again *forward rates are expected to stay the same, no matter what the yield curve does*

$$E_t\left(f_{t+1}^{(n)}\right) - f_t^{(n)} = f_t^{(n)} - f_t^{(n)} = 0$$

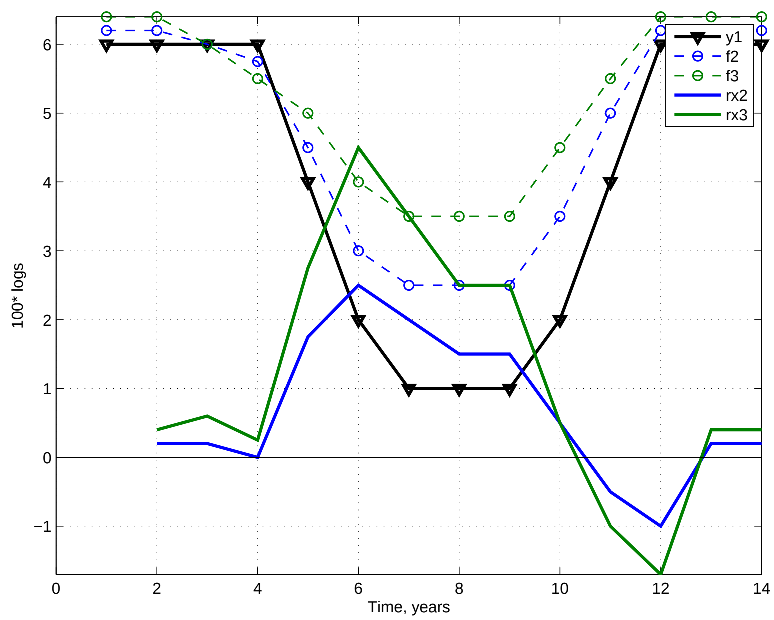
Note: he FB 0-1 result implies $E_t\left(y_{t+1}^{(1)}\right) = y_t^{(1)}$, while the long run regression is close to $y_{t+n-1}^{(1)} - y_t^{(1)} = 1(f_t^{(n)} - y_t^{(1)})$. In the example, the forecasts are 1% and 3.25%, which are very different! The point here is that Fama and Bliss's one year regression coefficients can't really be exactly one and zero. To be consistent with the longer-run coefficients, we need the one year slope coefficients to be a bit below one.

2. (a) $f_t^{(2)} = y_{t+1}^{(1)}; f_t^{(3)} = y_{t+2}^{(1)};$

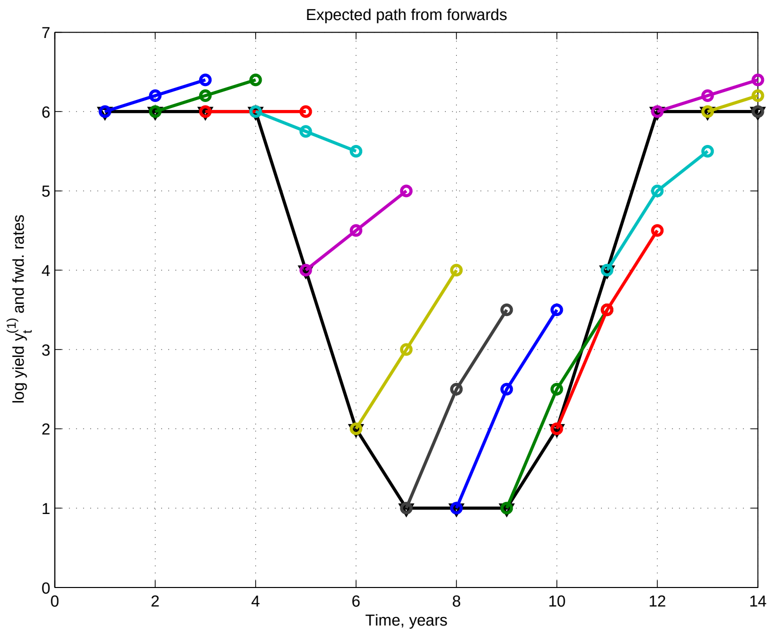



The forward rates move down well in advance of the recession. As we saw in the data, the forward rates are above spot rates even as rates decline.

(b) I calculated prices from forward rates, $p_t^{(1)} = -y_t^{(1)}; p_t^{(2)} = -y_t^{(1)} - f_t^{(2)}; p_t^{(3)} = -y_t^{(1)} - f_t^{(2)} - f_t^{(3)};$, then $rx_{t+1}^{(n)} = p_{t+1}^{(n-1)} - p_t^{(n)} - y_t^{(1)}$. You see the pattern, a strong positive ex-post rate of return going in to the recession and big losses when interest rates rise sharply on the other end. If people know what's going to happen these are expected returns.




(c)




These are the expectations that justify the forward curve. If the recession were a once off surprise, and most of the time interest rates followed the forecasts, this would make sense. The puzzle is that the forward rates suggest quick end to the recession, and instead the recession drags on. .One of two things must be true: Either recessions really are surprises, and our view that they predictably last a few years is wrong (the forward curve view) or there are big risk premiums early in the recession, and that drives the forward curve up even though everyone knows a recession is coming.

**Part II computer**

1

Fama Bliss Regression Coefficients, 19640831 20131231

rx(n) (t+1) = a + b [ f (n) (t) - y(1) (t)]

| maturity | a | b | s(a) | s(b) | R2 | R2unadj |
|---|---|---|---|---|---|---|
| 2 | 0.15 | 0.83 | 0.28 | 0.26 | 0.11 | 0.12 |
| 3 | 0.04 | 1.15 | 0.53 | 0.34 | 0.13 | 0.14 |
| 4 | -0.19 | 1.40 | 0.74 | 0.41 | 0.16 | 0.16 |
| 5 | 0.11 | 1.14 | 1.00 | 0.45 | 0.09 | 0.09 |

y(1)(t+n) -y(1)(t) = a + b [ f (n) (t) - y(1) (t)]

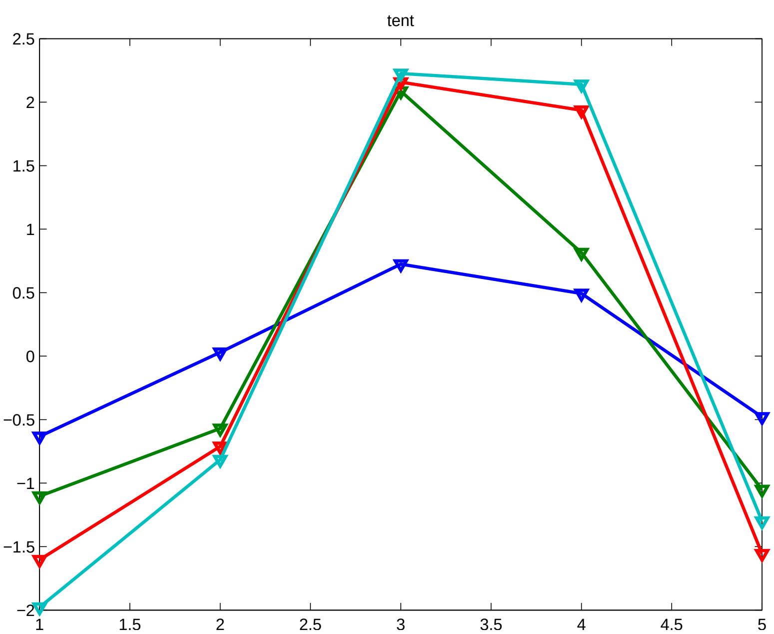
| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | -0.15 | 0.17 | 0.28 | 0.26 | 0.00 | 0.01 |
| 2 | -0.57 | 0.51 | 0.61 | 0.32 | 0.05 | 0.05 |
| 3 | -1.06 | 0.77 | 0.75 | 0.24 | 0.12 | 0.12 |
| 4 | -1.28 | 0.87 | 0.83 | 0.16 | 0.16 | 0.16 |

note: standard errors use Hansen-Hodrick GMM correction for overlapping data

2.

Returns on all forwards, 19640831 20131231

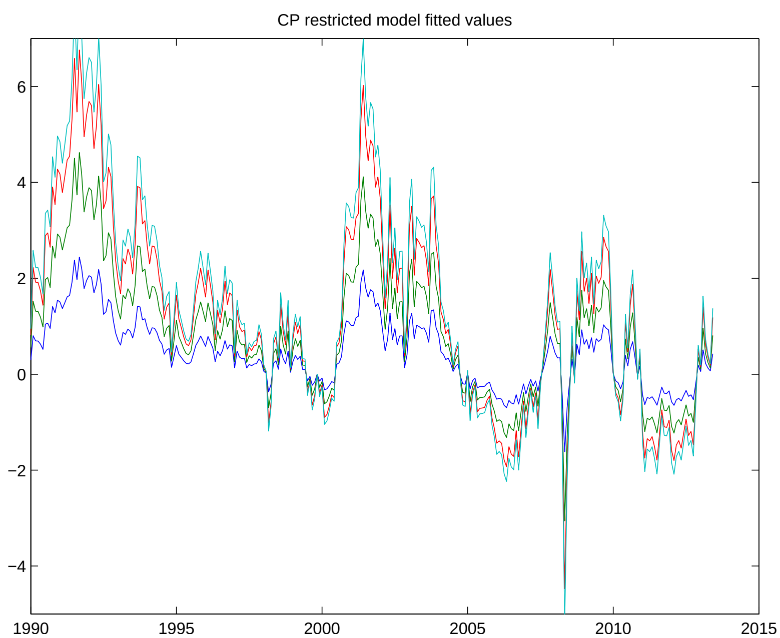
| matur | const | y1 | f(2) | f(3) | f(4) | f(5) | R2 |
|---|---|---|---|---|---|---|---|
| 2.00 | -0.81 | -0.63 | 0.03 | 0.72 | 0.49 | -0.48 | 0.21 |
| 3.00 | -1.14 | -1.10 | -0.57 | 2.09 | 0.81 | -1.05 | 0.22 |
| 4.00 | -1.65 | -1.60 | -0.71 | 2.16 | 1.94 | -1.56 | 0.25 |
| 5.00 | -2.31 | -1.98 | -0.82 | 2.23 | 2.14 | -1.30 | 0.23 |



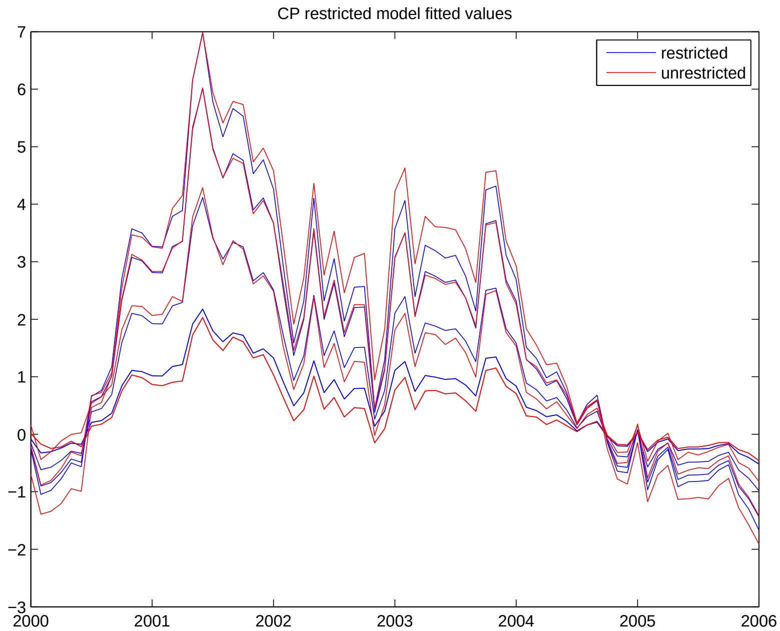
It is a lot less "tenty" but still quite similar across maturities." The $R^2$ are down to 0.21-0.25 because of the poor performance during the financial crisis, as explained in class. But a good deal better than Fama Bliss' 0.15 or so.

b) Here are my plots. They should convince you that the single-factor restriction is very very good.


c) and the same plot for the exact single-factor model.




I blew up a section and plotted the restricted (part b, which really is a one factor model) and unrestricted on the same graph. If it were perfect, these would be on top of each other. I rate it as darn good.



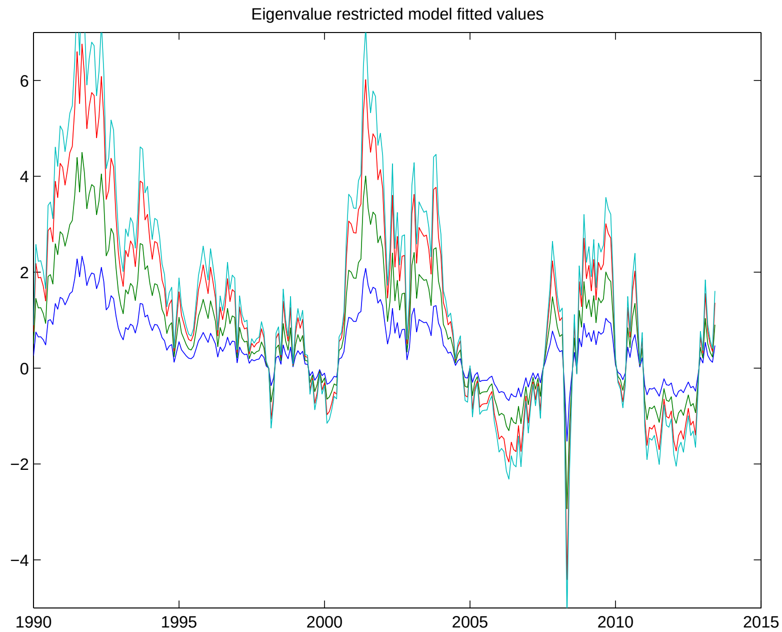

I put the plot first so you could compare. Here are the estimates

```
 Cochrane Piazzesi Single Factor Model
    matur    const     y(1)     f(2)     f(3)     f(4)     f(5)       R2
     av     -1.48    -1.33    -0.52     1.80     1.35    -1.10     0.23

   b(2)     b(3)     b(4)     b(5)
   0.45     0.85     1.25     1.45
```

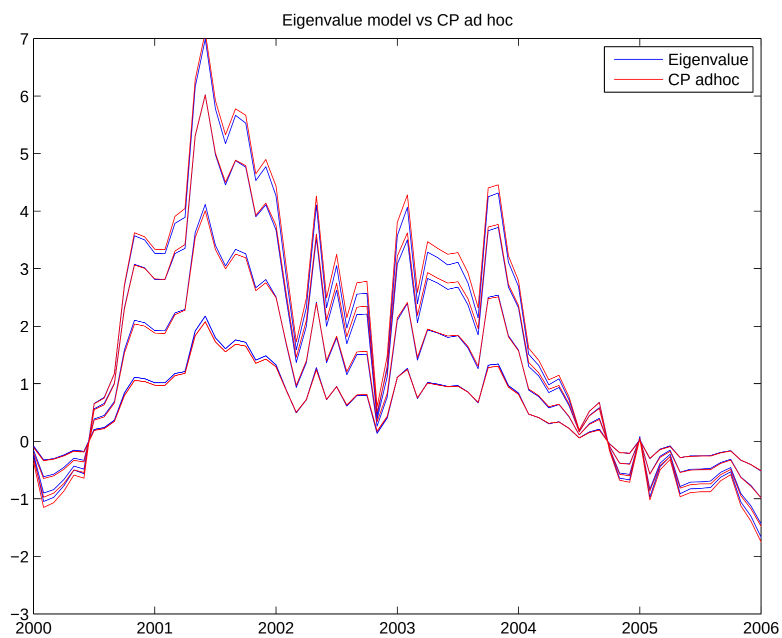
d). The code is ridiculously simple – a point here.

```
[Q,L] = eig(cov(fit(beg:T,:)));

q = Q(:,end);
factor = fit*q;
etrfit = factor*q';

figure;
plot(dates(beg:T),etrfit(beg:T,:));
```

Here is the plot



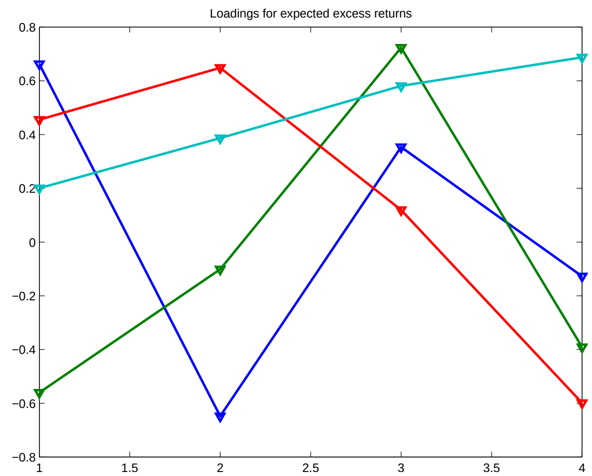

I can't tell it from previous graphs. Here is a zoom view of this one-factor model vs. the CP one-factor model. Above, I compared the CP model to the unrestricted expected returns. Here is a comparison of the eigenvalue model to the CP model. THey are almost exactly the same.

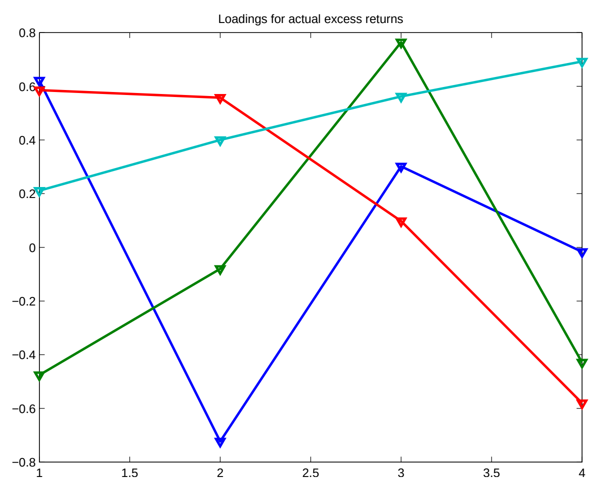



I investigated a little further (!) and here is a comparison of the factor model for *expected* excess returns with the factor model for *actual* excess returns. Here are the eigenvalues, first as standard deviations of the principal components $\sqrt{\lambda_i}$ and then as percent of variance explained by each component $100 \times \lambda_i / \sum_{i=1}^{4} \lambda_i$

```
return factor standard deviations and percent of variance
    0.22    0.29    0.79    8.07
    0.07    0.13    0.94   98.86
expected return factor standard deviations and percent of variance
    0.16    0.18    0.41    3.90
    0.17    0.21    1.07   98.54 \bigskip
```

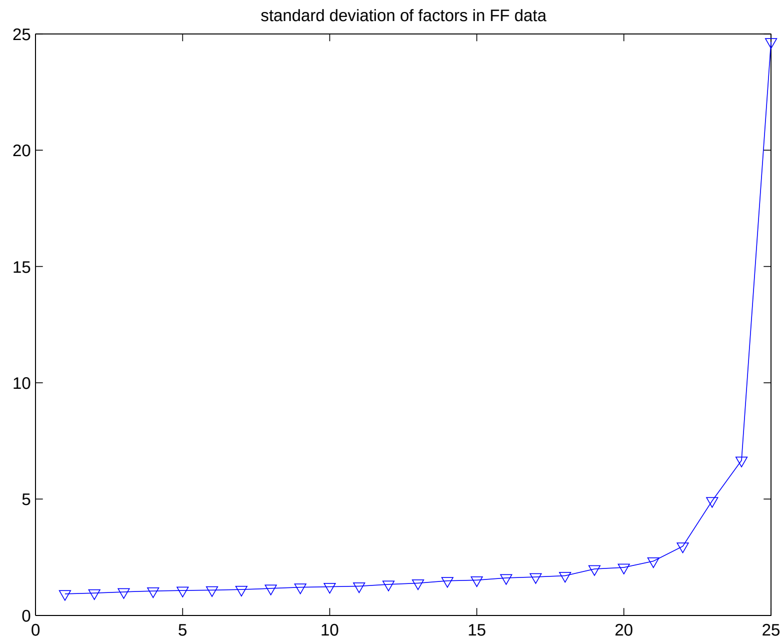
and here are the plots of the columns of Q – the weights and loadings




You get the picture: *returns and expected returns have nearly identical factor analyses.* When you think about it, this makes a lot of sense of course. If *returns* had a perfect single factor model, then *expected* returns would have to have a single factor model too! It's not necessarily true that second, third, and so on factors in returns need to match those in expected returns. But they do. That means that *each factor of returns is forecast by the same factor of expected returns.* Neat.

3. Here are the standard deviations of the factors. There's a huge first factor, 3 smaller factors, and then a tail. A picture like this usually means take the first 4 factors and ignore the rest, especially if the first four have interpretable loadings and the rest don't.



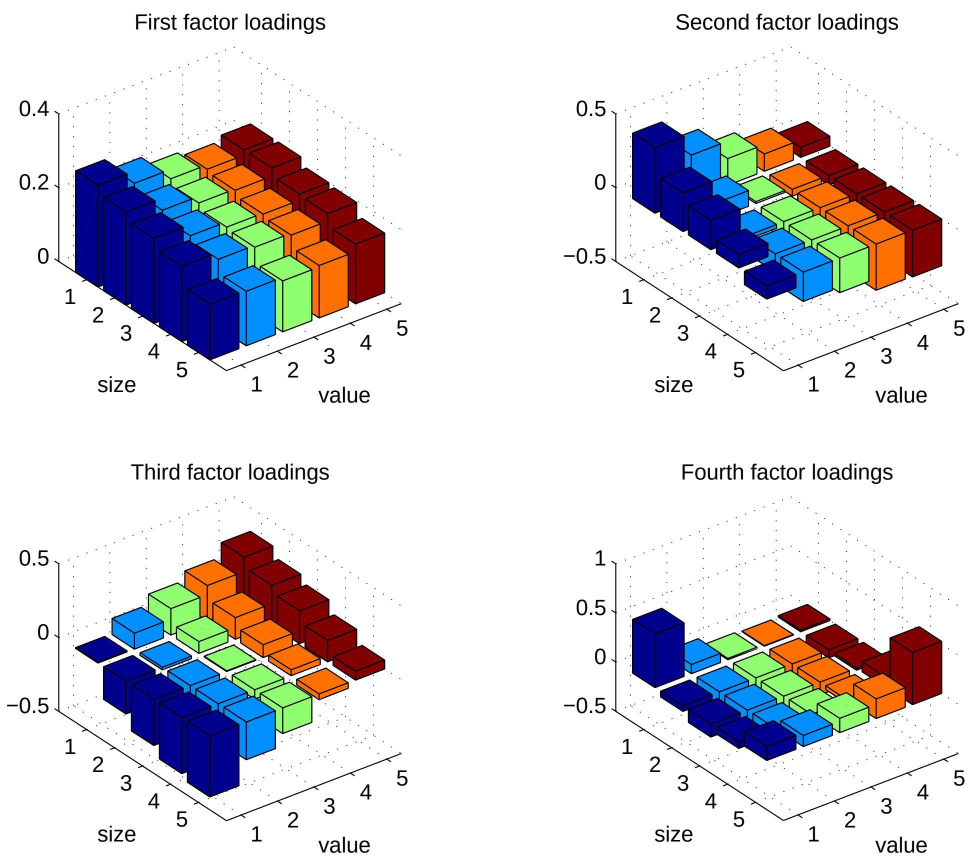

Next, the loadings. See the plots below.

The first factor is a "level" factor, moving everyone in the same direction. It's not flat – it moves small firms more than big firms, so it contains a bit of size as well as rmrf. The loadings are more like the single-regression loadings on the market that we saw when evaluating the CAPM on these portfolios. With interpretation 2 – loadings as the method for constructing the factor portfolio – we see that the first factor puts more weight on small firms. (Even equal weights would make the first factor the equally weighted market, not the value weighted market. It's quite interesting that the first factor is not the value-weighted market, which would have a hugely larger loading on the big portfolios.)

The second and third factors combine size and value. The second one is a bit more size-oriented. It moves small firms up and large firms down, but it also has a bit of value. The third factor has a bit more value – it moves the growth stocks down and the value stocks up – but it also has a size tilt. Clearly, between the two factors we have size and value factors. Using interpretation 2, we form these factor pretty much by combining small - big and the third as value - growth.

You're seeing one subtle issue here in factor models. The factors are separately "identified" by our attempt to order the variances. Since the variances of factors 2,3,4 are pretty much the same, the program has a hard time telling them apart.

The fourth factor moves small growth and large value up, and everyone else slightly down. This is really cool – small growth and large value are the Fama French puzzles in terms of alpha and look, they move together. Once again, high returns correspond to comovement, not to arbitrage.




How do mean returns line up with these covariances? I just fired up tsregress again.

The betas are the same as the columns of Q. Here are the alphas. I start with mean returns, and verify the usual pattern, higher mean with small size and especially value. The 1 factor model leaves substantial alphas. You have to watch the vertical axis here, as it changes. The alphas also have a pattern, they are larger for value stocks. There is a size pattern too – alphas are negative for small growth but positive for small value.

c) The "expected relationship" is that betas should equal the columns of Q. Thus, we don't have to make any more plots – we just did it.

```
sample
      194701      201312
betas 25 portfolios on 3 factors
    0.0059    0.4432    0.2754
    0.1078    0.2992    0.2431
    0.1787    0.1803    0.2177
    0.2409    0.1172    0.2054
    0.3385    0.0664    0.2202
   -0.2211    0.2602    0.2522
   -0.0177    0.1058    0.2193
    0.0787    0.0118    0.2013
    0.1467   -0.0447    0.1978
    0.2711   -0.0639    0.2205
   -0.3120    0.1975    0.2302
```

```
   -0.1048   -0.0263    0.2000
    0.0043   -0.1134    0.1853
    0.0866   -0.1664    0.1827
    0.2217   -0.1657    0.1995
   -0.3787    0.0884    0.2039
   -0.1663   -0.1229    0.1851
   -0.0631   -0.1968    0.1801
    0.0402   -0.1976    0.1748
    0.1439   -0.2625    0.1953
   -0.4132   -0.0886    0.1502
   -0.2558   -0.1993    0.1476
   -0.1730   -0.2346    0.1380
   -0.0379   -0.3126    0.1419
    0.0589   -0.3174    0.1616

columns of Q -- note same as betas
    0.0059    0.4432    0.2754
    0.1078    0.2992    0.2431
    0.1787    0.1803    0.2177
    0.2409    0.1172    0.2054
    0.3385    0.0664    0.2202
   -0.2211    0.2602    0.2522
   -0.0177    0.1058    0.2193
    0.0787    0.0118    0.2013
    0.1467   -0.0447    0.1978
    0.2711   -0.0639    0.2205
   -0.3120    0.1975    0.2302
   -0.1048   -0.0263    0.2000
    0.0043   -0.1134    0.1853
    0.0866   -0.1664    0.1827
    0.2217   -0.1657    0.1995
   -0.3787    0.0884    0.2039
   -0.1663   -0.1229    0.1851
   -0.0631   -0.1968    0.1801
    0.0402   -0.1976    0.1748
    0.1439   -0.2625    0.1953
   -0.4132   -0.0886    0.1502
   -0.2558   -0.1993    0.1476
   -0.1730   -0.2346    0.1380
   -0.0379   -0.3126    0.1419
    0.0589   -0.3174    0.1616
```

ii) Here are the $R^2$ and a comparison to FF. The $R^2$ are all *better* than FF's! The factor analysis solves the problem, how do you form three factors that maximize R2, so it had better improves!

```
3 factor r2
    0.9269    0.9418    0.9498    0.9557    0.9566
    0.9559    0.9436    0.9424    0.9406    0.9426
    0.9575    0.9294    0.9295    0.9354    0.9193
    0.9559    0.9367    0.9310    0.9164    0.9033
    0.8821    0.9014    0.8406    0.8790    0.8065

4 factor R2
    0.9712    0.9436    0.9499    0.9557    0.9567
```

| | | | | |
|---|---|---|---|---|
| 0.9565 | 0.9523 | 0.9581 | 0.9516 | 0.9443 |
| 0.9615 | 0.9500 | 0.9487 | 0.9418 | 0.9194 |
| 0.9570 | 0.9442 | 0.9377 | 0.9170 | 0.9064 |
| 0.8908 | 0.9074 | 0.8511 | 0.8977 | 0.9038 |

FF R2

| | | | | |
|---|---|---|---|---|
| 0.8609 | 0.9124 | 0.9306 | 0.9426 | 0.9468 |
| 0.9377 | 0.9366 | 0.9366 | 0.9396 | 0.9468 |
| 0.9450 | 0.9047 | 0.8979 | 0.9057 | 0.8973 |
| 0.9331 | 0.8924 | 0.8834 | 0.8870 | 0.8694 |
| 0.9438 | 0.9030 | 0.8503 | 0.8949 | 0.8266 |

iii) In terms of alphas. The 3 factor model does better than the FF3 factor model. Most of the alphas are small. Here is a table. Most of these are smaller than the FF3F counterparts. There is still a huge negative alpha for small growth and large value, as with FF3F. The rmse alpha is a bit better than FF, and a lot better with the "small growth, large value" factor.

- The factor procedure produces better $R^2$ pretty much by construction. Better alphas are an economic result. It means that the biggest unexplained *mean* from the 3 factor model corresponds to the biggest unexplained *variance.* It didn't have to come out that way.

3 factor alphas

| | | | | |
|---|---|---|---|---|
| -0.3687 | 0.0476 | 0.0423 | 0.1633 | 0.1728 |
| -0.1353 | 0.0190 | 0.1189 | 0.0565 | 0.0206 |
| 0.0482 | 0.0657 | 0.0118 | 0.0213 | 0.0227 |
| 0.1623 | -0.0733 | -0.0051 | -0.0219 | -0.1463 |
| 0.1579 | 0.0001 | 0.0238 | -0.1323 | -0.1846 |

rmse alpha
0.1219

4 factor alphas

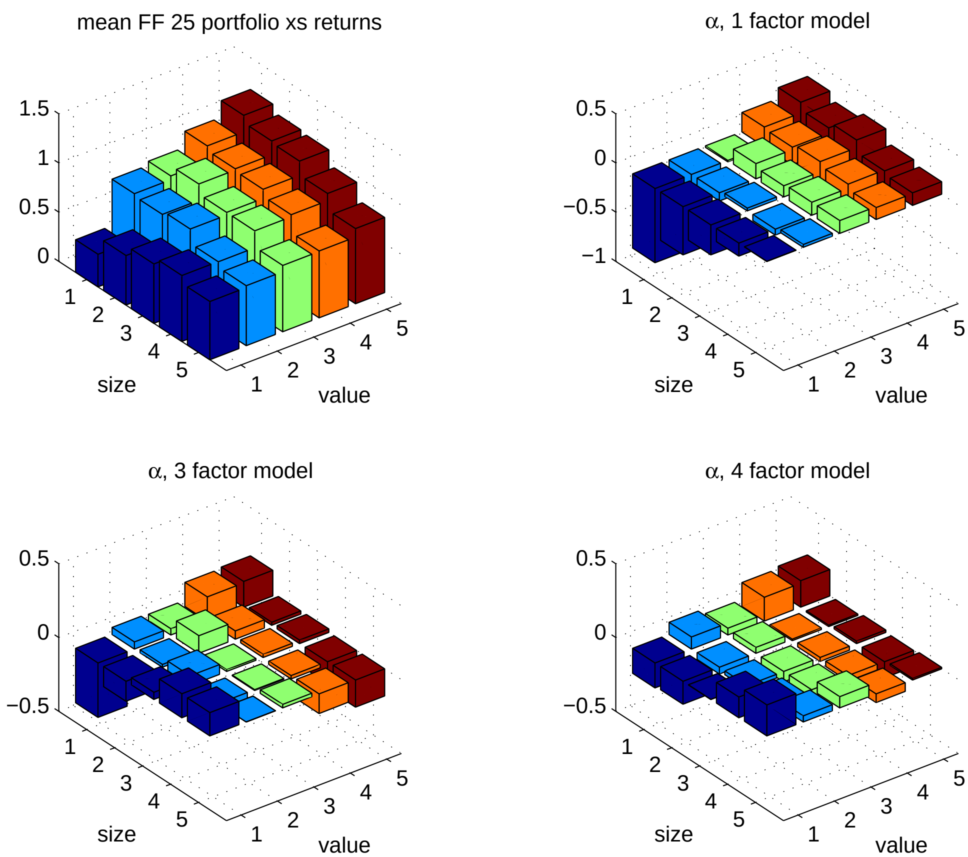
| | | | | |
|---|---|---|---|---|
| -0.1686 | 0.0816 | 0.0464 | 0.1615 | 0.1786 |
| -0.1562 | -0.0457 | 0.0394 | -0.0095 | -0.0089 |
| -0.0001 | -0.0256 | -0.0706 | -0.0264 | 0.0151 |
| 0.1395 | -0.1256 | -0.0538 | -0.0364 | -0.1088 |
| 0.2102 | 0.0411 | 0.0768 | -0.0587 | 0.0096 |

rmse alpha
0.0973

FF alphas

| | | | | |
|---|---|---|---|---|
| -0.5150 | -0.0533 | -0.0206 | 0.1115 | 0.1295 |
| -0.2272 | -0.0251 | 0.0988 | 0.0377 | -0.0028 |
| -0.0402 | 0.0474 | 0.0162 | 0.0301 | 0.0355 |
| 0.0847 | -0.0803 | 0.0070 | -0.0128 | -0.1187 |
| 0.1342 | -0.0021 | 0.0166 | -0.1301 | -0.1872 |

rmse alpha
0.1365




The 4th factor model loads on the large value and small growth, interestingly enough. Then, when we add it to the mix, it eliminates the large value-small growth puzzle. (In the same way that a momentum factor eliminates the momentum puzzle.)

Again, there is no guarantee that covariances will explain alphas. That's a result, not a mathematical certainty. If it were not true of course there would be high Sharpe ratio opportunities. Thus it's wonderful to see each factor in turn beat down the alpha puzzles of the previous factor model.

Disclaimer: Of course we should be cautious in the use of too many factors. They may not be stable out of sample. Also, the size factor had questionable economics, the value factor only had FF's speculations about human capital in depressed industries, and momentum has no economics. I have no hint of the economics behind a small growth - large value factor. Thus, you should probably view it as the momentum factor, an ad hoc device that may be useful for performance evaluation, but still on shaky ground for fundamental asset pricing.