Business 35904 **John H. Cochrane**

Problem Set 4

1. In this problem we will contrast two approaches to optimal portfolios in an interesting example. The question is, find the optimal portfolio for an investor with a habit, minimum subsistence level, drawdown limit, leverage, etc.:

$$\max E\left[\frac{(W_T-h)^{1-\gamma}}{1-\gamma}\right]$$

in the standard environment

$$\frac{dS}{S}=\mu dt+\sigma dz;\ \frac{dB}{B}=rdt.$$

In the portfolio readings, I solved this from the complete markets approach. A quick review: We write the discount factor

$$\frac{d\Lambda}{\Lambda}=-rdt-\frac{\mu-r}{\sigma^2}\sigma dz=-rdt-x\sigma dz;\ x\equiv\frac{\mu-r}{\sigma^2}.$$

Solving forward, this expression gives

$$\begin{aligned}
\ln\left(\frac{S_T}{S_0}\right) &= \left(\mu-\frac{1}{2}\sigma^2\right)T+\sigma\int_{s=0}^T dz_s\\
\ln\left(\frac{\Lambda_T}{\Lambda_0}\right) &= -\left(r+\frac{1}{2}x^2\sigma^2\right)T-x\sigma\int_{s=0}^T dz_s\\
&= \left[-\left(r+\frac{1}{2}x^2\sigma^2\right)+x\left(\mu-\frac{1}{2}\sigma^2\right)\right]T-x\ln\left(\frac{S_T}{S_0}\right).
\end{aligned}$$

Then, we find the optimal wealth at time T by

$$\begin{aligned}
(W_T-h)^{-\gamma} &= \lambda\frac{\Lambda_T}{\Lambda_0}\\
W_T &= \lambda^{-\frac{1}{\gamma}}\left(\frac{\Lambda_T}{\Lambda_0}\right)^{\frac{1}{\gamma}}+h.
\end{aligned}$$

Evaluating the wealth constraint to eliminate the Lagrange multiplier,

$$W_0=E\left(\frac{\Lambda_T}{\Lambda_0}W_T\right)=\lambda^{-\frac{1}{\gamma}}E\left(m^{1-\frac{1}{\gamma}}\right)+he^{-rT},$$

we obtain

$$W_T=\left(W_0-he^{-rT}\right)\frac{\left(\frac{\Lambda_T}{\Lambda_0}\right)^{-\frac{1}{\gamma}}}{E\left(\left(\frac{\Lambda_T}{\Lambda_0}\right)^{\frac{\gamma-1}{\gamma}}\right)}+h$$

and, taking the expectation in the denominator,

$$W_T=\left(W_0-he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\left(\frac{S_T}{S_0}\right)^{\alpha}+h;\quad \alpha=\frac{x}{\gamma}=\frac{1}{\gamma}\frac{\mu-r}{\sigma^2}$$

This is a very sensible answer. First and foremost, the investor guarantees the payoff h. Then, wealth left over after buying a bond that guarantees h, $\left(W_0-he^{-rT}\right)$ is invested in a way that takes on more risk the lower γ and thus the higher α, becoming more sensitive to S_T.

Good, but how do I *implement* that answer? What do I actually *buy? What is the dynamic trading strategy that the habit investor should follow to get this payoff by a combination of stocks and bonds?*

a) *Implementation of a complete-markets answer.* Our first approach to answering this question stays in the complete markets tradition. Given W_T, we can find the dynamic strategy in two steps: First find the value of the investor's wealth at any date prior to T by

$$W_t = E_t\left(\frac{\Lambda_T}{\Lambda_t} W_T\right).$$

Then, take dW_t using Ito's lemma, and match its terms to find w_t in

$$\frac{dW_t}{W_t} = rdt + w_t\left(\frac{dS_t}{S_t} - rdt\right)$$

This w_t is then the weight in the risky asset in the portfolio that gets you to W_T. Do it. Express your answer in two, equivalent ways:

i) First, the investor puts he^{-rT} into bonds, to make sure he can cover the habit h, and then he invests $W_0 - he^{rT}$ into a constantly rebalanced portfolio of stocks and bonds. By taking $\frac{d\left[W_t - he^{-r(T-t)}\right]}{\left[W_t - he^{-r(T-t)}\right]}$ you will be able to find the weight of that portfolio in the risky asset.

ii) Second, just think of the whole portfolio as a dynamically rebalanced portfolio of stocks and bonds, with a "risk aversion" that rises as wealth declines. By taking $\frac{dW_t}{W_t}$ you will be able to find the shares of that portfolio.

Express both sets of weights i) and ii) w_t in terms of W_t. (I.e. not in terms of S_t. It's possible to express in terms of S_t, but I didn't do it, and the answer in terms of W_t is very intuitive.)

b) *Implementation by value function.* The "standard" way to set up this problem is to write the value function, as a reminder,

$$V(W_t, t; h) = \max_{w_t} E_t\left[V(W_{t+\Delta}, t+\Delta)\right] \text{ s.t. } ...$$

$$0 = \max_{w_t} E_t\left[dV(W_t, t)\right] \text{ s.t. } \frac{dW_t}{W_t} = r + w_t\left(\frac{dS}{S} - rdt\right)$$

$$0 = \max_{w_t} E_t\left[V_t dt + V_W dW + \frac{1}{2}V_{WW}dW^2\right] \text{ s.t. } ...$$

$$0 = \max_{w_t}\left[\frac{V_t}{W} + V_W\left[r + w_t(\mu - r)\right] + \frac{1}{2}WV_{WW}w_t^2\sigma^2\right]$$

$$w_t = \left(-\frac{V_W}{WV_{WW}}\right)\left(\frac{\mu - r}{\sigma^2}\right) \tag{1}$$

$$0 = r + \frac{V_t}{WV_W} + \frac{1}{2}\left(-\frac{V_W}{WV_{WW}}\right)\frac{(\mu - r)^2}{\sigma^2} \tag{2}$$

Well great, now we have to solve this partial differential equation backward from the boundary condition

$$V(W_T, T; h) = \frac{(W_T - h)^{1-\gamma}}{1-\gamma}.$$

The usual method is, guess the form, then find the undetermined coefficients. So, you have two choices

i) Guess the form of the value function. The final value

$$V(T) = \frac{(W_T - h)^{1-\gamma}}{1-\gamma}$$

is a place to start. Guess what it looks like for $t < T$ up to undetermined coefficients. (Hint: put $e^{a(T-t)}$ in the right place.) Put your guess into (2) to find the undetermined coefficients. If it doesn't work, refine your guess.

ii) Guessing the form isn't easy. We can find the value function from the complete market answer, however, which will let us guess what form to use. The value function is, after all,

$$V(t) = E_t\left[V(T)\right] = E_t\left[\frac{(W_T - h)^{1-\gamma}}{1-\gamma}\right]. \tag{3}$$

We derived above

$$W_T - h = \left(W_0 - he^{-rT}\right) e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T} \frac{S_T^\alpha}{S_0^\alpha}$$

So by taking the expectation in (3) you can find the actual value function. Take the expectation in (3), and then check that your value function does in fact satisfy (2).

By either method i) or ii), find the value function. Then use (1) to find the portfolio weights rule w_t as a function of W_t, etc., which is what we're looking for.

c) *Implementation by options.* Rather than dynamically trade, find a portfolio of stock, bond, and European call options, bought or written at date 0, expiring at date T, that generate the optimal portfolio without any intermediate trading. This problem involves some technical issues if $\alpha < 1$, so you need only solve the case $\alpha > 1$.

Hint: Suppose you bought 1 (well, really $1dX$) call option at each strike X. What would the payoff from this strategy be as a function of S_T? Now, go back and figure out how many options at each X, $g(X)dX$, you need to buy to get a general payoff $f(S_T)$. The restriction $\alpha > 1$ means you have $f(0) = \lim_{S_T \to 0} kS_T^\alpha = 0$ and $f'(0) = \lim_{S_T \to 0} \alpha k S_T^{\alpha-1} = 0$, which will come in handy. If then you want to tackle the case $0 < \alpha < 1$, and thus $f(0) \neq 0$, $f'(0) \neq 0$, you'll be ready, but you don't have to do this case. I did it by thinking of a limit in which each element of the sequence was well behaved.

Make a plot of $g(X)$ and the corresponding W_T for $\alpha = 2.5, 2, 1.5$ and 1.1, and explain the pattern of options across strikes intuitively. Hint: The first hint gave you the answer for $\alpha = 2$.

2. We have sort of beat to death the standard continuous time-power portfolio theory, and shown how it leads to a mean-variance efficient portfolio choice. In this problem you'll explore the other standard utility function specifications, each of which also leads to mean-variance portfolio choice in one-period problems with no outside income.

Our investor starts with wealth W_0 and considers investing in N stocks with excess return R_{t+1}^e $(N \times 1)$and a bond with return R_t^f. The investor consumes tomorrow only, so his objective is

$$\max_{\{w\}} Eu(c_{t+1}) = Eu(W_{t+1}) = Eu(R_{t+1}^p W_0) = Eu\left[(R_t^f + w'R_{t+1}^e)W_0\right]$$

a) Start by characterizing the mean variance efficient portfolio so you'll know it when you find it. (We've done this before, and you can go look up the formulas if you prefer. Or rederive them here.) Find a formula for w and $R^{ep} = w'R^e$ in terms of the mean and covariance matrix of returns, $\min_w \sigma^2(w'R^e)$ s.t. $E(w'R^e) = \mu$. Also prove that $\min_w \sigma^2\left[w'R^e\right] + \mu^2 = \min_w E\left[(w'R^e)^2\right]$ s.t. $E(w'R^e) = \mu$ gives the same result, giving you a formula in terms of the second moment matrix of returns. It will also be useful to remind yourself that $R^{e*} = E(R^{e\prime})E(R^e R^{e\prime})^{-1}R^e$ is on the mean-variance efficient frontier.

b) Find the first-order condition for the portfolio maximization (take the derivative with respect to w) and relate it to our friend $0 = E(mR^e)$

c) Suppose utility is quadratic $u(c_{t+1}) = -\frac{1}{2}(c^* - c_{t+1})^2$. Show that the investor chooses a mean-variance efficient portfolio.

d) Suppose utility is exponential

$$u(c) = -e^{-\alpha c}; u'(c) = \alpha e^{-\alpha c}$$

where α is the "coefficient of absolute risk aversion." Assume that excess returns are normally distributed. Show that the investor chooses a mean-variance efficient portfolio. To solve this case *first* take the expectation of utility, using the fact that if x is normally distributed $E(e^x) = e^{E(x)+\frac{1}{2}\sigma^2(x)}$. *Then* take the derivative with respect to w and set it to zero. Sometimes it's easier to do it this way rather than find the first-order condition $0 = E\left[u'(R^f + wR^e)R^e\right]$ then take expectations, and then take derivative with respect to w.

(Note: Pay attention. This functional form is very popular in trading models because it gives a linear demand for stocks. Then adding up demands and intersecting supply and demand is very easy. Look for it in lots of trading and microstructure models.)

3) In class, we talked in class about the intuition of whether risk premiums should be at the long or short end of the yield curve. I offered two reasons why many models predict (counterfactually) a downward-sloping risk premium. First, I argued that long term (real) bonds are riskfree assets for long horizon investors. Fine, you said, but that doesn't prove that short term bonds have positive or negative risk premiums, and we need to see some equations. Second, I argued that $E(R^e) = cov(R^e, \Delta c)\gamma$; when interest rates go up long term bond prices go down. Since interest rates go up when the economy recovers, that suggests that $cov(R^e, \Delta c) < 0$ for long term bonds, that they should have a negative excess return relative to short term bonds. But then we realized that the covariance is also an endogenous variable in bond pricing models, and we were hungry to link the expected return idea to the slope of the yield curve.

Let's figure this all out in the context of the single-factor Vasicek model.

If you recall, we posit a latent state variable which follows an AR(1),

$$\left(y_{t+1}^{(1)} - \delta\right) = \phi\left(y_t^{(1)} - \delta\right) + \varepsilon_{t+1}.$$

and then the discount factor follows

$$\gamma \Delta c_{t+1} = -\ln M_{t+1} = y_t^{(1)} + \frac{1}{2}\lambda^2\sigma_\varepsilon^2 + \lambda\varepsilon_{t+1}$$

By adding the $-\gamma\Delta c_{t+1}$ part, we will be able to think about the economic determinants of risk premiums in this model.

We price assets as usual.

$$P_t^{(1)} = E_t\left(M_{t+1}1\right) = E_t e^{-y_t^{(1)} - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 - \lambda\varepsilon_{t+1}} = e^{-y_t^{(1)}}$$

so the latent state variable is "revealed" by bond prices as $y_t^{(1)} = -p_t^{(1)}$.

Now we can see that ε_{t+1} are interpreted as interest rate shocks. In this model, ex-post consumption shocks are perfectly correlated with interest rate shocks, $\left(E_{t+1} - E_t\right)\Delta c_{t+1} = \lambda\varepsilon_{t+1}/\gamma$. And the only reason interest rates change is because expected consumption growth changes. So, we are going to end up linking the slope of the term structure and risk premiums in bond returns to properties of the consumption process.

As a reference, here are the bond pricing formulas. From $p_t^{(n)} = \log E_t\left(e^{m_{t+1}+p_{t+1}^{(n-1)}}\right)$ we have

$$p_t^{(n)} = -n\delta + A_n - B_n\left(y_t^{(1)} - \delta\right)$$

where

$$\begin{aligned} B_n &= 1+\phi B_{n-1}; \text{or } B_n = B_{n-1} + \phi^{n-1} \\ B_1 &= 1; B_2 = (1+\phi); B_3 = (1+\phi+\phi^2); \\ B_n &= (1+\phi+\phi^2+..+\phi^{n-1}) = \frac{1-\phi^n}{1-\phi} \end{aligned}$$

and

$$A_n = A_{n-1} + \left[\frac{1}{2}B_{n-1}^2 + B_{n-1}\lambda\right]\sigma_\varepsilon^2;\ A_1 = 0.$$

Hence,

$$\begin{aligned} f_t^{(n)} &= p_t^{(n-1)} - p_t^{(n)} = \delta - (A_n - A_{n-1}) + (B_n - B_{n-1})\left(y_t^{(1)} - \delta\right) \\ f_t^{(n)} &= \delta - \left[\frac{1}{2}B_{n-1}^2 + B_{n-1}\lambda\right]\sigma_\varepsilon^2 + \phi^{n-1}\left(y_t^{(1)} - \delta\right) \end{aligned}$$

a) Use $1 = E_t(M_{t+1}R_{t+1}) = Ee^{m_{t+1}+r_{t+1}}$ to find $E_t(r_{t+1})$ in terms of $cov(r_{t+1}, \varepsilon_{t+1})$ and, separately, in terms of $cov(r_{t+1}, (E_{t+1} - E_t)\,\Delta c_{t+1})$ for a generic normally distributed log return r_{t+1} in this environment. These will provide a useful intuitive formula. They look like formulas you've seen before, but there is an extra $1/2\sigma^2$ term. They also will help to clarify the role of λ.

b) Find the ex-post return $r_{t+1}^{(n)} = p_{t+1}^{(n-1)} - p_t^{(n)} = y_t^{(1)} + () + ()\varepsilon_{t+1}$. The two terms are expected excess return and the sensitivity of return to the interest rate shock. If interest rates $y_t^{(1)}$ rise unexpectedly, does the ex-post return rise or fall? (The point – we are showing here how the model *derives* the covariance of returns with ε_{t+1}, rather than taking that covariance as a primitive the way we usually do with equities.)

c) Verify that your formula for ex-post return satisfies the $E_t(r_{t+1})$ equation you derived in part a.

d) Now, let us look at the return risk premium defined as

$$E_t r_{t+1}^{(n)} - y_t^{(1)}$$

and the forward rate risk premium defined as

$$f_t^{(n)} - E_t y_{t+n-1}^{(1)}$$

Find these two objects, and their relationship. Abstracting from the $1/2\sigma^2$ term, does $\lambda > 0$ imply higher or lower expected returns on long term bonds? Does $\lambda > 0$ imply that forward rates are higher or lower than expected future spot rates (i.e. does the forward curve slope up or down for $\lambda > 0$?)

e) So, now our job is to think about the economics of λ. Is it plausible that $\lambda > 0$ or $\lambda < 0$? In class, I argued for $\lambda > 0$, because I said when consumption growth rises, interest rates will rise too. I meant that as a rough description of data. However, in this model, interest rates come from expected consumption growth, so my supposition that $\lambda > 0$ means I am saying that when consumption growth rises ε_{t+1}, then expected consumption growth $E_{t+1}\Delta c_{t+2}$ also rises – consumption growth is positively serially correlated. (In the model there is no inflation. It's possible that my view of the data means that expected inflation rises when consumption growth rises, and we would need to add inflation and an inflation risk premium to address that.) This observation is another nice one for using models, as it ties down the covariance of consumption growth with interest rate shocks to the serial correlation of consumption growth. Let's explore this connection a little further, and incidentally learn something about the time series model for consumption. Reparameterizing so things are clearer, our model is

$$\frac{\left(y_{t+1}^{(1)}-\delta\right)}{\gamma}=\phi\frac{\left(y_{t}^{(1)}-\delta\right)}{\gamma}+\frac{1}{\gamma}\varepsilon_{t+1}.$$

$$\Delta c_{t+1}=\frac{1}{\gamma}\left(y_{t}^{(1)}-\delta\right)+\frac{\delta}{\gamma}+\frac{1}{2\gamma}\lambda^{2}\sigma_{\varepsilon}^{2}+\frac{\lambda}{\gamma}\varepsilon_{t+1}$$

or, with

$$\begin{aligned}
x_t &= \frac{1}{\gamma}\left(y_{t}^{(1)}-\delta\right)\\
v_{t+1} &= \frac{\lambda}{\gamma}\varepsilon_{t+1},\\
a &= \frac{\delta}{\gamma}+\frac{1}{2}\gamma\sigma_{v}^{2}
\end{aligned}$$

we can write

$$x_{t+1}=\phi x_t+\frac{1}{\lambda}v_{t+1} \tag{4}$$

$$\Delta c_{t+1}=a+x_t+v_{t+1} \tag{5}$$

In this form you see the standard finance time-series model: x_t carries movements in $E_t\Delta c_{t+1}$; then Δc_{t+1} has an additional shock. That shock may be correlated with the shock to x_t, in this (simple) case it is perfectly correlated. I reparameterized it this way to focus on the *consumption* process and consumption shocks. (We often write return processes this way, with a slow time-varying expected return.)

In general, with the shocks to x and the shock to c imperfectly correlated, the history of x gives more information than the history of c. Since the shocks are perfectly correlated (and an invertibility condition holds), however, consumption growth forecasts here are the same as in the univariate representation. So let's transform to the univariate representation to see what λ does.

Finally, something for you to do: Find the univariate (Wold) representation for consumption corresponding to (4)-(5). Hint: It's an ARMA(1,1). Substitute $x_t=(1-\phi L)^{-1}v_t/\lambda$ in (5) to find it.

f) Now, start by finding the case in which consumption growth is uncorrelated over time. You hold σ_v^2 constant and change λ. (An ARMA(1,1) $(1-\phi L)\Delta c_t=a+(1-\theta L)v_t$ reverts to iid when $\theta=\phi$ and the roots cancel.) Find the corresponding parameters of the original model, and find the return and forward rate risk premium for this case (define "risk premium" to be the terms multiplying λ). (It will be easier to think of this in terms of a limit.)

g) Now, suppose $\infty>\lambda>0$. Is consumption positively or negatively serially correlated? At all horizons? Find the impulse-response function for this case. This is the assumption about consumption implicit in the downward-sloping yield curve case.

Note: This problem brings up the issues, not the final answer! As you saw, this model tightly links ex-post consumption changes (risk exposure) to expected consumption changes and hence real interest rate changes. More general models loosen that up. This is the *way* to think about the economic intuition, but not the final intuition itself.