

Problem Set 11

1. Let's actually solve a few very simple classic portfolio problems. Our investor starts with wealth W_t and considers investing in a single stock (the index) with return R_{t+1} and a bond with return R_t^f . Denote $R_{t+1}^e = R_{t+1} - R_t^f$. The investor consumes tomorrow only, so his objective is

$$\begin{aligned} \max_{\{w\}} E[u(W_{t+1})] &= E[u(R_{t+1}^p W_t)] \\ &= E\left\{u\left[(wR_{t+1}^e + R_t^f)W_t\right]\right\} \end{aligned}$$

- (a) Find the first-order condition for this maximization (take the derivative with respect to w) and express it in the form $E[u'(\cdot)R_{t+1}^e] = 0$. Notice that this is our friend $p = E(mx)$, and we are now determining $c_{t+1} = W_{t+1} = R_{t+1}^p W_t$ given the properties of returns, not determining $E(R_{t+1}^e)$ given the properties of consumption.
- (b) Suppose utility is quadratic $u(c_{t+1}) = -\frac{1}{2}(c^* - c_{t+1})^2$ where c^* is a number. Show that the optimal portfolio weight depends only on the mean and variance of the excess return. (You will get a formula that depends on $E(R^e)$ and $E(R^{e2}) = E(R^e)^2 + \sigma^2(R^e)$.)
- (c) Suppose utility is exponential

$$u(c) = -e^{-\alpha c}; u'(c) = \alpha e^{-\alpha c}$$

α is the “coefficient of absolute risk aversion.” Find the optimal portfolio weight w and show that w depends only on mean and variance $E(R_{t+1}^e)$ and $\sigma^2(R_{t+1}^e)$.

To solve this case *first* take the expectation of utility, assuming that R_{t+1}^e is normally distributed, and using the fact that if x is normally distributed $E(e^x) = e^{E(x) + \frac{1}{2}\sigma^2(x)}$. Then take the derivative with respect to w and set it to zero. Sometimes it's easier to do it this way rather than find the first-order condition $0 = E[u'(R^f + wR^e)R^e]$ then take expectations, and then take derivative with respect to w .

2. This is a very simple version of the power-lognormal “real” portfolio problem that we are solving.

The investor wants to maximize $E[u(W_T)] = E\left(W_T^{1-\gamma}\right)$, by investing from time zero to time T starting with initial wealth W_0 . The investor can put money in a stock whose log return is normally distributed with mean arithmetic return μ and standard deviation σ , and a bond that earns $r^f = 0$. We will use the following fact: If the investor holds a constantly rebalanced portfolio with weight w in the stock, wealth at time T is also lognormally distributed,

$$\frac{W_T}{W_0} = e^{[w\mu - \frac{1}{2}w^2\sigma^2]T + w\sigma\sqrt{T}\varepsilon}; \varepsilon \sim N(0, 1) \quad (1)$$

(I derive this below FYI)

- (a) Use $E[e^x] = e^{E(x) + \frac{1}{2}\sigma^2(x)}$ to find $E\left[W_T^{1-\gamma}\right]$.
- (b) Now, find the optimal portfolio. Maximize the last expression by setting the derivative of $E\left[W_T^{1-\gamma}\right]$ with respect to w to zero, and solve for w .
- (c) What is the effect of T in your optimal formula? Should a long-horizon investor hold more stocks because stocks are safer in the long run (with these assumptions!)

- (d) Even if returns are independent over time, a friend argues, stocks are safer for long-run investors. True, the one year return has a 20% standard deviation, but 25-year average returns $\frac{1}{T}(r_1 + r_2 + \dots + r_T)$ have a $20\%/\sqrt{25} = 4\%$ standard deviation, while the average return is the same for any horizon. He also points out that the Sharpe ratio $\frac{E(r_1+r_2+\dots+r_T)}{\sigma(r_1+r_2+\dots+r_T)}$ rises with the square root of horizon. Is the implication that stocks are better for long-run investors in this situation right?
- (e) What considerations are left out of this problem that might tilt the optimum towards stocks for longer horizon investors?

(Note, not necessary to do the problem. Where did (1) come from? To do this you need a little dz and dt . S_t is stock price, and the stock's instantaneous arithmetic returns follow

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t.$$

A bond has risk free rate r^f . (dz_t is a normally distributed random variable with mean 0 and variance dt . It is the continuous time version of the ε_t we have been using to describe time series. This just says $r_{t+1} = \frac{S_{t+1}-S_t}{S_t} = \mu + \sigma\varepsilon_{t+1}$.)

Now, if you put weight w into the stock, wealth W evolves as

$$\frac{dW}{W} = w \frac{dS}{S} + (1-w)r^f dt$$

$$\frac{dW}{W} = w(\mu dt + \sigma dz_t) + (1-w)r^f dt$$

Using Ito's lemma, (second order expansion and $dz^2 = dt$)

$$d \log W = \left[w\mu + (1-w)r^f - \frac{1}{2}w^2\sigma^2 \right] dt + w\sigma dz_t$$

$$\log W_T - \log W_0 = \left[w(\mu - r^f) + r^f - \frac{1}{2}w^2\sigma^2 \right] T + w\sigma \int_{t=0}^T dz_t$$

and with $\int_{t=0}^T dz_t = \sqrt{T}\varepsilon, \varepsilon \sim N(0, 1)$,

$$\frac{W_T}{W_0} = e^{[r^f + w(\mu - r^f) - \frac{1}{2}w^2\sigma^2]T + w\sigma\sqrt{T}\varepsilon}$$

I specialized to $r^f = 0$.)

3. Note: This is an especially good problem. You are considering investing in two managers, and of course the market index. You have a mean-variance objective with risk aversion $\gamma = 2$. Your assessment of the market portfolio is a mean $E(R^{em}) = 8\%$, volatility $\sigma(R^{em}) = 20\%$. You run CAPM regressions for the two managers

$$R_t^{ei} = \alpha_i + \beta_i R_t^{em} + \varepsilon_t^i$$

with result $\beta_1 = 2, \beta_2 = 2$; $\sigma(\varepsilon) = 10\%$ for both managers, and the residuals ε have correlation -0.5 . You believe $\alpha_1 = -0.3\%$, $\alpha_2 = 1.2\%$.

- (a) Find the optimal allocation to the market index and to the two managers. (Hint: Be careful about units, i.e. should you express 10% as 10, 1.10, 0.10?) Express the answer in terms of a weight w_m on the excess market return R_t^{em} and weights w_α on "portable alpha" or beta-hedged portfolios for each of the two managers - ($R_t^{ei} - \beta_i R_t^{em} = \alpha_i + \varepsilon_t^i$, the manager's return less his beta times the market return) - i.e. write the optimal portfolio return in the form, $R^{ep} = w_m R^{em} + w_1 (R^{e1} - \beta_1 R_t^{em}) + w_2 (R^{e2} - \beta_2 R_t^{em})$ and find the weights w_m, w_1, w_2 .

- (b) Find the weights in terms of the actual zero-cost investments, i.e. R_t^{em} and R_t^{e1}, R_t^{e2} . i.e. $R^{ep} = \hat{w}_m R^{em} + \hat{w}_1 R^{e1} + \hat{w}_2 R^{e2}$. (I used hats because these may be different from part a)
- (c) Find the weights in terms of actual investments, i.e. $R^p = w^{f*} R^f + w_m^* R^m + w_1^* R^1 + w_2^* R^2$ (again, * because these might be different from the weights in a and b.
- (d) Compare the three sets of weights. Which are the same, which are different?
- (e) In this example, you end up investing a positive amount with a manager who has negative alpha. How is this possible? (Get ready to market your negative alpha hedge fund!) Hint: If the managers' ε were uncorrelated with each other, could it happen?
4. The new neurofinance researchers are able to connect your brain to electrodes and conclusively measure your risk aversion $\gamma = 2$. Your best guess is that the mean annual premium is 8% with volatility 20%. Returns are independent over time and you are rich enough that you will never need a job.
- (a) What should your allocation to stocks be?
- (b) In fact you don't really know what the mean return is. Reflecting on it, your uncertainty (standard error) about the mean return is 5 percentage points, $\sigma[E(R^e)] = 0.05$. How does this consideration affect your allocation to stocks?
- (c) Same question, but now you have a 10 year horizon. Since returns are iid, the mean return and variance of return scale with horizon. Use $E(R^{10 \text{ year}}) = 10 \times E(R^{1 \text{ year}})$, $\sigma^2(R^{10 \text{ year}}) = 10 \times \sigma^2(R^{1 \text{ year}})$. Your estimation uncertainty is still 5% on an annual basis, so $\sigma[E(R^{10 \text{ year}})] = 10 \times \sigma[E(R^{1 \text{ year}})]$.
- (d) Compare b and c. Does parameter uncertainty matter more or less at longer horizons?

5. After much hard work, you create a "small business" factor, a portfolio of assets that has 90% correlation with shocks to the profits of small and privately-held businesses. Alas, after running regressions of your portfolio return on rrmf, smb, hml, you find the alpha of your factor return is zero

$$R^{\text{factor}} = 0 + b \times rrmf_t + h \times hml_t + s \times smb_t + \varepsilon^{\text{factor}}; R^2 = 0.4$$

This means we do not need your new factor to price assets, and the *expected* returns of your factor can be achieved by the combination of FF 3 factors. Might a fund based on your new factor be interesting to investors nonetheless? How would you use your investment strategy to form the fund?

6. You find a great new return-forecasting factor, whether the National League ($NL = 1$) or American League ($NL = 0$) wins the world series, i.e. that b and R^2 are large in forecasting the market return,

$$R_{t+1} = a + b \times NL_t + \varepsilon_{t+1}.$$

However, you find that *current* stock market returns are not correlated with who wins the world series,

$$R_{t+1} = a + 0 \times NL_{t+1} + \delta_{t+1}$$

and the correlation of R_{t+1} or δ_{t+1} with ε_{t+1} is zero. How does the presence of NL affect the optimal portfolio allocation between stocks and bonds? (There are two components, "market timing" and "optimal hedging." Comment on both.)

7. Using standard statistics, long term bonds look like terrible investments. The expected return is essentially no different from that of short term bonds. For the purposes of this problem, assume that the long and term bonds have exactly the same expected return, $E(R_{t+1}^{\text{long}}) = 0$. Long term bonds have much more interest rate risk $\sigma^2(R_{t+1}^{\text{long}}) = 10\%$ and of course $\sigma(R_{t+1}^{\text{short}}) = 0$. Therefore,

$w = \frac{1}{\gamma} \frac{E_t(R^e)}{\sigma^2(R^e)}$ says you should put zero weight in long-term bonds. Why might it make sense for an investor to hold long term bonds anyway? Use formulas from the class/notes for optimal one-period portfolio weights to make your point, as well as any other arguments that matter (Hint: Do expected returns of long term bonds vary over time? What's a good state variable for the investment opportunities of long term bonds?)