Problem Set 4 Answers

1)

a) In the end, we're looking for

$$\frac{d\left[W_t - he^{-r(T-t)}\right]}{\left[W_t - he^{-r(T-t)}\right]} = (1-\alpha)rdt + \alpha\frac{dS_t}{S_t}$$

This suggests writing the portfolio as an investment he^{-rT} in the riskless asset, then investing $W_0 - he^{-rT}$ in the risky asset as if you were an investor with no wealth. The other answer is

$$\frac{dW_t}{W_t} = rdt + \left\{\left[\frac{W_t - he^{-r(T-t)}}{W_t}\right]\frac{1}{\gamma}\right\}\frac{\mu-r}{\sigma^2}\left(\frac{dS_t}{S_t} - rdt\right)$$

here, the investor has a "risk aversion" that rises as wealth approaches the habit.

$$rra = \gamma\frac{W_t}{W_t - he^{-r(T-t)}}$$

The derivation: This is a bit repetitive, because we are doing the same algebra at t that we did at $t = 0$ to evaluate the λ constraint. We should have done that once and for all to answer both questions. But having been given the solution in terms of W_0 we can somewhat kludgeily repeat the same algebra to find W_t :

$$W_t = E_t\left(\frac{\Lambda_T}{\Lambda_t}W_T\right) = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\left(\frac{S_t}{S_0}\right)^\alpha E_t\left[\frac{\Lambda_T}{\Lambda_t}\left(\frac{S_T}{S_t}\right)^\alpha\right] + he^{-r(T-t)}$$

$$E_t\left[\frac{\Lambda_T}{\Lambda_t}\left(\frac{S_T}{S_t}\right)^\alpha\right] = E\left[e^{\left(-r-\frac{1}{2}x^2\sigma^2\right)(T-t)-x\sigma\int_{s=t}^T dz_s}e^{\alpha\left(\mu-\frac{1}{2}\sigma^2\right)T+\sigma\alpha\int_{s=t}^T dz_s}\right]$$

$$= E\left[e^{\left[-r-\frac{1}{2}x^2\sigma^2+\frac{x}{\gamma}\mu-\frac{1}{2}\frac{x}{\gamma}\sigma^2\right](T-t)+\left(\frac{1}{\gamma}-1\right)x\sigma\int_{s=t}^T dz_s}\right]$$

$$= e^{\left(-r-\frac{1}{2}x^2\sigma^2+\frac{x}{\gamma}\mu-\frac{1}{2}\frac{x}{\gamma}\sigma^2+\frac{1}{2}\left(\frac{1}{\gamma}-1\right)^2 x^2\sigma^2\right)(T-t)}$$

$$= e^{\left[-r-\frac{1}{2}\gamma^2\left(\frac{x}{\gamma}\right)^2\sigma^2+\frac{x}{\gamma}\mu-\frac{1}{2}\frac{x}{\gamma}\sigma^2+\frac{1}{2}(\gamma-1)^2\left(\frac{x}{\gamma}\right)^2\sigma^2\right](T-t)}$$

$$= e^{\left(-r+\frac{x}{\gamma}\mu+\left(\left(\frac{1}{2}-\gamma\right)\left(\frac{x}{\gamma}\right)-\frac{1}{2}\right)\frac{x}{\gamma}\sigma^2\right)(T-t)}$$

$$= e^{\left(-r+\frac{x}{\gamma}r+\frac{x}{\gamma}\sigma^2\frac{(\mu-r)}{\sigma^2}+\left(\left(\frac{1}{2}-\gamma\right)\left(\frac{x}{\gamma}\right)-\frac{1}{2}\right)\frac{x}{\gamma}\sigma^2\right)(T-t)}$$

$$= e^{\left(-r+\frac{x}{\gamma}r+\left(+\gamma\left(\frac{x}{\gamma}\right)+\left(\frac{1}{2}-\gamma\right)\left(\frac{x}{\gamma}\right)-\frac{1}{2}\right)\frac{x}{\gamma}\sigma^2\right)(T-t)}$$

$$= e^{\left(\frac{x}{\gamma}-1\right)\left(r+\frac{1}{2}\frac{x}{\gamma}\sigma^2\right)(T-t)}$$

so,

$$W_t = E_t\left(\frac{\Lambda_T}{\Lambda_t}W_T\right) = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\left(\frac{S_t}{S_0}\right)^\alpha e^{-(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)(T-t)} + he^{-r(T-t)}$$

$$W_t - he^{-r(T-t)} = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)t}\left(\frac{S_t}{S_0}\right)^\alpha$$

Well, duh, you say. The investor puts $\left(W_0 - he^{-rT}\right)$ into the usual stock/bond split, and lets it grow. OK, but we're here really to confirm that the method works.

Now, let's follow through and find the actual portfolio by taking dW_t,

$$\frac{d\left[W_t - he^{-r(T-t)}\right]}{\left[W_t - he^{-r(T-t)}\right]} = (1-\alpha)\left(r + \frac{1}{2}\alpha\sigma^2\right)dt + \alpha\frac{dS_t}{S_t} + \frac{1}{2}\alpha(\alpha-1)\frac{dS_t^2}{S_t^2}$$

$$= (1-\alpha)\left(r + \frac{1}{2}\alpha\sigma^2\right)dt + \alpha\frac{dS_t}{S_t} + \frac{1}{2}\alpha(\alpha-1)\sigma^2 dt$$

$$\frac{d\left[W_t - he^{-r(T-t)}\right]}{\left[W_t - he^{-r(T-t)}\right]} = (1-\alpha)rdt + \alpha\frac{dS_t}{S_t}$$

This suggests writing the portfolio as an investment he^{-rT} in the riskless asset, then investing $W_0 - he^{-rT}$ in the risky asset as if you were an investor with no wealth. If you weren't clever enough to write it that way, however, you'd (after a lot of algebra) end up writing it as a dynamic trading strategy.

$$\frac{d\left[W_t - he^{-r(T-t)}\right]}{\left[W_t - he^{-r(T-t)}\right]} = (1-\alpha)rdt + \alpha\frac{dS_t}{S_t}$$

$$dW_t - rhe^{-r(T-t)}dt = \left[W_t - he^{-r(T-t)}\right](1-\alpha)rdt + \left[W_t - he^{-r(T-t)}\right]\alpha\frac{dS_t}{S_t}$$

$$\frac{dW_t}{W_t} = \frac{he^{-r(T-t)}}{W_t}rdt + \left[1 - \frac{he^{-r(T-t)}}{W_t}\right](1-\alpha)rdt + \left[1 - \frac{he^{-r(T-t)}}{W_t}\right]\alpha\frac{dS_t}{S_t}$$

$$\frac{dW_t}{W_t} = rdt + \left[1 - \frac{he^{-r(T-t)}}{W_t}\right]\alpha\left(\frac{dS_t}{S_t} - rdt\right)$$

$$\frac{dW_t}{W_t} = rdt + \left[\frac{W_t - he^{-r(T-t)}}{W_t}\right]\frac{1}{\gamma}\frac{\mu-r}{\sigma^2}\left(\frac{dS_t}{S_t} - rdt\right)$$

so in this description, there is a time-varying decision rule where the investor changes weights as W_t changes.

b) The central answers are

$$V(W_t, t; h) = \frac{1}{1-\gamma}e^{(1-\gamma)\left[r + \frac{1}{2}\gamma\alpha^2\sigma^2\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{(1-\gamma)}$$

$$w_t = \left(-\frac{V_W}{WV_{WW}}\right)\left(\frac{\mu-r}{\sigma^2}\right) = \left(\frac{1}{\gamma}\frac{W_t - he^{-r(T-t)}}{W_t}\right)\left(\frac{\mu-r}{\sigma^2}\right)$$

The derivation. I'm not smart enough to guess, so I cranked out the value function.

$$V(t) = E_t[V(T)] = E_t\left[\frac{(W_T - h)^{1-\gamma}}{1-\gamma}\right]$$

$$= \frac{1}{1-\gamma}E_t\left\{\left[\left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r + \frac{1}{2}\alpha\sigma^2\right)T}\frac{S_T^\alpha}{S_0^\alpha}\right]^{1-\gamma}\right\}.$$

We derived above

$$W_T - h = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r + \frac{1}{2}\alpha\sigma^2\right)T}\frac{S_T^\alpha}{S_0^\alpha}$$

and

$$W_t - he^{-r(T-t)} = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r + \frac{1}{2}\alpha\sigma^2\right)t}\frac{S_t^\alpha}{S_0^\alpha},$$

so we can write the term inside E_t as

$$\left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\frac{S_T^\alpha}{S_0^\alpha} = \left(W_t - he^{-r(T-t)}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)(T-t)}\frac{S_T^\alpha}{S_t^\alpha}$$

Now, we're ready to take the expectation,

$$V(t) = \frac{1}{1-\gamma}E_t\left\{\left[\left(W_t - he^{-r(T-t)}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)(T-t)}\frac{S_T^\alpha}{S_t^\alpha}\right]^{1-\gamma}\right\}$$

$$V(t) = \frac{1}{1-\gamma}\left[e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)(T-t)}\left(W_t - he^{-r(T-t)}\right)\right]^{1-\gamma}E_t\left\{\left[\frac{S_T^\alpha}{S_t^\alpha}\right]^{1-\gamma}\right\}$$

$$V(t) = \frac{1}{1-\gamma}\left[e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)(T-t)}\left(W_t - he^{-r(T-t)}\right)\right]^{1-\gamma}E_t\left\{e^{\left(\mu-\frac{\sigma^2}{2}\right)(1-\gamma)\alpha(T-t)+(1-\gamma)\alpha\sigma\int_{s=t}^T dz_s}\right\}$$

$$V(t) = \frac{\left(W_t - he^{-r(T-t)}\right)^{1-\gamma}}{1-\gamma}e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)(1-\gamma)(T-t)}e^{\left(\mu-\frac{\sigma^2}{2}\right)(1-\gamma)\alpha(T-t)+\frac{1}{2}(1-\gamma)^2\alpha^2\sigma^2(T-t)}$$

$$V(t) = \frac{\left(W_t - he^{-r(T-t)}\right)^{(1-\gamma)}}{1-\gamma}e^{(1-\gamma)\left[(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)+\left(\mu-\frac{\sigma^2}{2}\right)\alpha+\frac{1}{2}(1-\gamma)\alpha^2\sigma^2\right](T-t)}$$

$$V(t) = \frac{1}{1-\gamma}e^{(1-\gamma)\left[(1-\alpha)r+\alpha\mu-\frac{1}{2}\alpha^2\sigma^2\gamma\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{(1-\gamma)}$$

$$V(t) = \frac{1}{1-\gamma}e^{(1-\gamma)\left[r+\frac{1}{2}\gamma\alpha^2\sigma^2\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{(1-\gamma)}$$

As a quick check note $V(T)$ is what it should be.

Now, we can check that this "guess" works,

$$V_t = -\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)e^{(1-\gamma)\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)(T-t)}\left(W_t - he^{-r(T-t)}\right)^{(1-\gamma)}$$

$$-rhe^{-r(T-t)}e^{(1-\gamma)\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)(T-t)}\left(W_t - he^{-r(T-t)}\right)^{-\gamma}$$

$$V_t = -e^{(1-\gamma)\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)(T-t)}\left(\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)\left(W_t - he^{-r(T-t)}\right)+rhe^{-r(T-t)}\right)\left(W_t - he^{-r(T-t)}\right)^{-\gamma}$$

$$V_t = -e^{(1-\gamma)\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)(T-t)}\left(\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)\left(W_t - he^{-r(T-t)}\right)+rhe^{-r(T-t)}\right)\left(W_t - he^{-r(T-t)}\right)^{-\gamma}$$

$$V_W = e^{(1-\gamma)\left[r+\frac{1}{2}\gamma\alpha^2\sigma^2\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{-\gamma}$$

$$\frac{V_t}{WV_W} = \frac{-e^{(1-\gamma)\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)(T-t)}\left[\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)\left(W_t - he^{-r(T-t)}\right)+rhe^{-r(T-t)}\right]\left(W_t - he^{-r(T-t)}\right)^{-\gamma}}{W_t e^{(1-\gamma)\left[r+\frac{1}{2}\gamma\alpha^2\sigma^2\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{-\gamma}}$$

$$= -\frac{\left(r+\frac{1}{2}\gamma\alpha^2\sigma^2\right)\left(W_t - he^{-r(T-t)}\right)+rhe^{-r(T-t)}}{W_t}$$

$$= -r - \frac{1}{2}\gamma\frac{\left(W_t - he^{-r(T-t)}\right)}{W_t}\alpha^2\sigma^2$$

$$V_{WW} = -\gamma e^{(1-\gamma)\left[r+\frac{1}{2}\gamma\alpha^2\sigma^2\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{-\gamma-1}$$

$$-\frac{WV_{WW}}{V_W} = \frac{\gamma W e^{(1-\gamma)\left[r+\frac{1}{2}\gamma\alpha^2\sigma^2\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{-\gamma-1}}{e^{(1-\gamma)\left[r+\frac{1}{2}\gamma\alpha^2\sigma^2\right](T-t)}\left(W_t - he^{-r(T-t)}\right)^{-\gamma}}$$

$$= \gamma\frac{W}{W_t - he^{-r(T-t)}}$$

9

And finally, does it satisfy the partial differential equation?

$$0 =? \frac{V_t}{WV_W} + r + \frac{1}{2}\left(-\frac{V_W}{WV_{WW}}\right)\frac{(\mu - r)^2}{\sigma^2}$$

$$0 = -r - \frac{1}{2}\gamma\frac{\left(W_t - he^{-r(T-t)}\right)}{W_t}\alpha^2\sigma^2 + r + \frac{1}{2}\left(\frac{1}{\gamma}\frac{W_t - he^{-r(T-t)}}{W_t}\right)\frac{(\mu - r)^2}{\sigma^2}$$

$$0 = -\frac{1}{2}\gamma\frac{\left(W_t - he^{-r(T-t)}\right)}{W_t}\alpha^2\sigma^2 + \frac{1}{2}\left(\frac{1}{\gamma}\frac{W_t - he^{-r(T-t)}}{W_t}\right)\alpha^2\gamma^2\sigma^2!$$

Yes! Could you have guessed this? It would have been enough to guess

$$V(t) = \frac{1}{1-\gamma}e^{a(T-t)}\left(W_t - he^{-r(T-t)}\right)^{(1-\gamma)}$$

and then find a.

The whole point of this was to find the portfolio weights, from the value function. We had them before, but for completeness,

$$w_t = \left(-\frac{V_W}{WV_{WW}}\right)\left(\frac{\mu - r}{\sigma^2}\right)$$

$$w_t = \left(\frac{1}{\gamma}\frac{W_t - he^{-r(T-t)}}{W}\right)\left(\frac{\mu - r}{\sigma^2}\right)$$

c)The hint: suppose you hold one (well, $1 \times dX$) call option of each strike. What payoff do you get at each S_T? Answer: At each S_T you get $S_T - X$ from each option at $X < S_T$, so your payoff is

$$\int_{X=0}^{S_T} 1(S_T - X)dX = (S_T^2 - \frac{S_T^2}{2}) = \frac{1}{2}S_T^2$$

Now, for the question at hand. There are several ways to do this; my solution method is surely not unique. First, of course, buy bonds he^{-rT} which provide h. Now we are left with the question, how to use the money left over $\left(W_0 - he^{-rT}\right)$ to buy options to give a standard power-utility investor type payoff,

$$W_T - h = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\frac{S_T^\alpha}{S_0^\alpha}$$

One way to do it, following the hint: Let $g(X)dX$ denote the number of call options between strike X and strike $X + dx$ that the investor buys (or, if $g < 0$,writes.) Then for any S_T, the payoff of this strategy sums up all in-the-money call options times their payoff, $\int_{X=0}^{S_T} g(X)(S_T - X)dX$. If we want to create a payoff function

$$f(S_T) = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\frac{S_T^\alpha}{S_0^\alpha}$$

we want to find $g(X)$ such that

$$\int_{X=0}^{S_T} g(X)(S_T - X)dX = f(S_T)$$

Taking the second derivative with respect to S_T,

$$\frac{d}{dS_T} : \int_{X=\varepsilon}^{S_T} g(X)dX + b = f'(S_T)$$

10

$$\frac{d}{dS_T} : g(S_T) = f''(S_T)$$

That's very intuitive, and recalls the result that $\partial^2 C/\partial X^2$ is the price of contingent claims to X. But,there are a lot of functions f with the same second derivative. $f(S_T) + a + bS_T$ (adding a bond and stock position) gives the same answer. What's going on? Let's work backwards: Suppose we use $g(X) = f''(X)$. What payoff does this give (what a and b?)

$$\int_{X=0}^{S_T} g(X)(S_T - X)dX = \int_{X=0}^{S_T} f''(X)(S_T - X)dX$$

$$= \int_{X=0}^{S_T} f''(X)S_T dX - \int_{X=0}^{S_T} f''(X)X dX$$

evaluating the second integral by parts with $d[Xf'(X)] = Xf''(X)dX + f'(X)dX$

$$= S_T[f'(S_T) - f'(0)] - [S_T f'(S_T) - 0f'(0)] + \int_{X=0}^{S_T} f'(X)dX$$

$$= -[S_T - 0]f'(0) + f(S_T) - f(0)$$

So, we get back $f(S_T)$ if $f(0) = f'(0) = 0$. If not, we need to include the stock (a lump investment in a call option with $X = 0$) and bond term as well as the options. So, now you know why I suggested you start with the case $\alpha > 1$ which has $f(0) = f'(0)$. (I presume most of you were pressed for time and just plowed on from $g(X) = f''(X)$, and you'll be ok if you did.)

For our $f(S_T) = kS_T^\alpha$, this cute identity reads

$$\int_{X=0}^{S_T} k\alpha(\alpha - 1)X^{\alpha-2}(S_T - X)dX = kS_T^\alpha$$

If you're still doubtful (I was)

$$k\alpha(\alpha - 1)S_T \int_{X=0}^{S_T} X^{\alpha-2}dX - k\alpha(\alpha - 1) \int_{X=0}^{S_T} X^{\alpha-1}dX = kS_T^\alpha$$

$$k\alpha S_T S_T^{\alpha-1} - k(\alpha - 1)S_T^\alpha = kS_T^\alpha$$

yes, it works for $\alpha > 1$. So, in sum,

*For $\alpha > 1$, to replicate the payoff*

$$W_T - h = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\frac{S_T^\alpha}{S_0^\alpha}$$

with options, the investor at time 0 puts he^{-rT} in the riskfree asset, and then buys call options of strikes X in the amount

$$g(X) = \left(W_0 - he^{-rT}\right)e^{(1-\alpha)\left(r+\frac{1}{2}\alpha\sigma^2\right)T}\alpha(\alpha - 1)\frac{X^{\alpha-2}}{S_0^\alpha}$$

Here is my plot. 2.0 is the "hint" case – a portfolio of options constant across strikes gives rise to S_T^2. In the 2.5 case, the payoff is more convex than S_T^2, so we need to buy more and more options at larger and larger strikes to generate the convexity in $f(S_T)$ The $\alpha = 1.5$ case is the opposite. Now we need call options with a declining function of strikes, because $f(S_T)$ is less convex. The limit $\alpha = 1$ is the limit $f(S_T) = S_T$, which is a lump investment in a call at $X = 0$. As you can see $g(X)$ is headed in that direction.
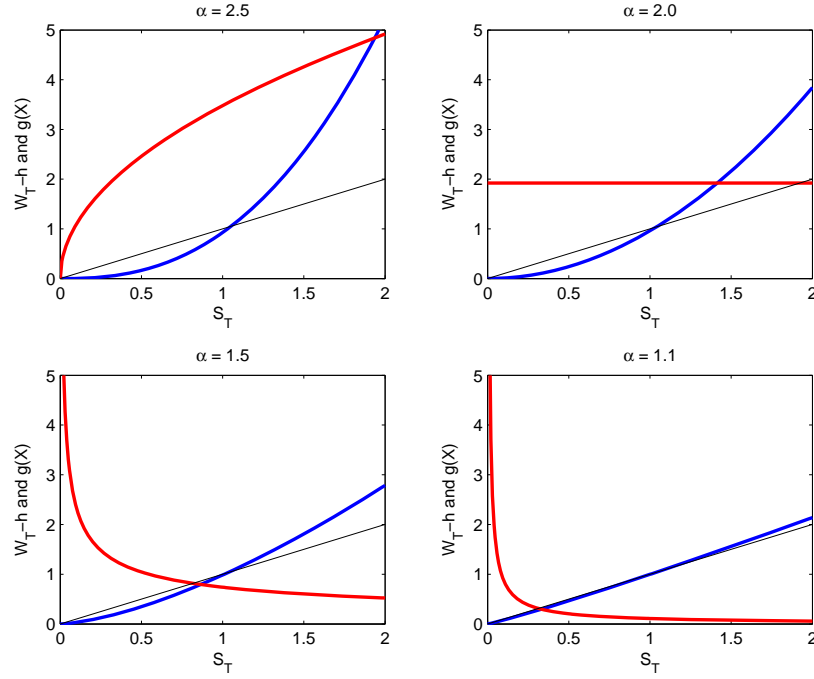
The cases for $0 < \alpha < 1$ pose a technical problem because not only do we have $f'(0) > 0$ which is pretty easy to solve (just add an investment in the stock to match $f'(0)$, but also $f'(0) = \infty$. Since options give a 45 degree payoff, you need a double infinity of options at $X = 0$, and then sell them off gradually for larger X. The technicalities of this case are not worth our time right now. (I worked on it a bit by starting at $X = \varepsilon$, and then taking the limit as $\varepsilon \to 0$.)

2.

a) This is old stuff.

$$\min_w \sigma^2\left(wR^e\right) \quad s.t.E(w'R^e) = \mu$$

$$\min_w w'\Sigma w \quad s.t.w'E = \mu$$

$$w = \lambda\Sigma^{-1}E$$

That is enough to characterize it, but if you want to put it in terms of μ,

$$\lambda E'\Sigma^{-1}E = \mu$$
$$w = \mu\left(E'\Sigma^{-1}E\right)^{-1}\Sigma^{-1}E$$
$$R^{ep} = \mu\left(E'\Sigma^{-1}E\right)^{-1}E'\Sigma^{-1}R^e.$$

Adding a constant to a maximization doesn't change anything:

$$\min_w \sigma^2\left(wR^e\right) + \mu^2 \quad s.t.E(w'R^e) = \mu$$

$$\min_w \sigma^2\left(wR^e\right) + \left[E\left(w'R^e\right)\right]^2 \quad s.t.E(w'R^e) = \mu$$

12

$$\min_{w} E\left[(wR^e)^2\right] \ \ s.t. E(w'R^e) = \mu$$

$$\min_{w} w'\Omega w \ \ s.t. E(w'R^e) = \mu$$

$$\Omega = E(R^e R^{e\prime}) = \Sigma + EE'$$

$$w = \mu\left(E'\Omega^{-1}E\right)^{-1}\Omega^{-1}E$$

$$R^{ep} = \mu\left(E'\Omega^{-1}E\right)^{-1}E'\Omega^{-1}R^e$$

and remember our friend R^{e*}, which is the special case $\mu = E'\Omega^{-1}E$.

$$R^{e*} = E\left(R^{e\prime}_{t+1}\right) E\left(R^e_{t+1}R^{e\prime}_{t+1}\right)^{-1} R^e_{t+1} = E'\Omega^{-1}R^e.$$

The second moment weights and covariance matrix weights are of course the same. I couldn't find a faster way to check but brute force works. I start with the result and work backwards. The proof starts at the bottom and works up.

$$\left(E'\Omega^{-1}E\right)^{-1}E'\Omega^{-1} = \left(E'\Sigma^{-1}E\right)^{-1}E'\Sigma^{-1}$$

$$\left(E'\Omega^{-1}E\right)^{-1}E' = \left(E'\Sigma^{-1}E\right)^{-1}E'\Sigma^{-1}\Omega$$

$$\left(E'\Omega^{-1}E\right)^{-1}E' = \left(E'\Sigma^{-1}E\right)^{-1}E'\Sigma^{-1}\left(\Sigma + EE'\right)$$

$$\left(E'\Omega^{-1}E\right)^{-1}E' = \left(E'\Sigma^{-1}E\right)^{-1}E'\Sigma^{-1}\Sigma + \left(E'\Sigma^{-1}E\right)^{-1}E'\Sigma^{-1}EE'$$

$$\left(E'\Omega^{-1}E\right)^{-1}E' = \left(E'\Sigma^{-1}E\right)^{-1}E' + E'$$

$$\left(E'\Omega^{-1}E\right)^{-1} = \left(E'\Sigma^{-1}E\right)^{-1} + I$$

$$I = \left(E'\Omega^{-1}E\right)\left(E'\Sigma^{-1}E\right)^{-1} + \left(E'\Omega^{-1}E\right)$$

$$\left(E'\Sigma^{-1}E\right) = \left(E'\Omega^{-1}E\right) + \left(E'\Omega^{-1}E\right)\left(E'\Sigma^{-1}E\right)$$

$$\left(E'\Sigma^{-1}E\right) = \left(E'\Omega^{-1}E\right) + \left(E'\Omega^{-1}E\right)\left(E'\Sigma^{-1}E\right)$$

$$\Sigma^{-1} = \Omega^{-1} + \Omega^{-1}EE'\Sigma^{-1}$$

$$I = \Omega^{-1}\Sigma + \Omega^{-1}EE'$$

$$\Omega = \Sigma + EE'$$

b) From lecture,

$$\max_{[w]} E\left[u(c_{t+1})\right]$$

$$c_{t+1} = W_{t+1} = R^p_{t+1}W_t$$

$$R^p_{t+1} = R^f_t + wR^e_{t+1}$$

$$\max_{[w]} E\left\{u\left[\left(R^f_t + w'R^e_{t+1}\right)W_0\right]\right\}$$

$$\frac{d}{dw} : E\left\{u'\left[\left(R^f_t + w'R^e_{t+1}\right)W_0\right]R^e_{t+1}\right\} = 0$$

c) Quadratic.

$$u(c) = -\frac{1}{2}(c^* - c)^2$$

$$u'(c) = c^* - c$$

$$E\left\{\left(c^* - \left(R_t^f + w'R_{t+1}^e\right)W_0\right)R_{t+1}^{e\prime}\right\} = 0$$

$$E\left\{\left(c^* - R_t^f W_0 + w'W_0 R_{t+1}^e\right)R_{t+1}^{e\prime}\right\} = 0$$

$$\left(c^* - R_t^f W_0\right)E\left(R_{t+1}^{e\prime}\right) - w'W_0 E\left(R_{t+1}^e R_{t+1}^{e\prime}\right) = 0$$

$$w' = \frac{\left(c^* - R_t^f W_0\right)}{W_0}E\left(R_{t+1}^{e\prime}\right)E\left(R_{t+1}^e R_{t+1}^{e\prime}\right)^{-1}$$

$$w'R_{t+1}^e = \frac{\left(c^* - R_t^f W_0\right)}{R^f W_0}R^f E\left(R_{t+1}^{e\prime}\right)E\left(R_{t+1}^e R_{t+1}^{e\prime}\right)^{-1}R_{t+1}^e = \frac{\left(c^* - R_t^f W_0\right)}{R^f W_0}R^f R_{t+1}^{e*}$$

We know R^{e*} is on the mean-variance frontier. Note that relative risk aversion is

$$\gamma = \frac{-cu''(c)}{u'(c)} = \frac{c}{(c^* - c)}$$

So, we can express

$$w = \frac{1}{\gamma}R^f R^{e*}$$

if we define

$$\gamma = \frac{-cu''(c)}{u'(c)} = \frac{R^f W}{(c^* - R^f W)}$$

as the local coefficient of risk aversion, evaluated at the point c that would be generated by putting it all in the risk free rate, $w = 0$. So, investors who are less risk averse invest more in stocks. This is not exactly the

$$w = \frac{1}{\gamma}\Sigma^{-1}\mu$$

formula that holds in continuous time, but it's pretty close.

d) Normal-exponential.

$$u(c) = -e^{-\alpha c}$$

$$E\left[-e^{-\alpha\left(R_t^f + w'R_{t+1}^e\right)W_0}\right] = -e^{-\alpha R_t^f W_0 - \alpha w'E\left(R_{t+1}^e\right)W_0 + \frac{1}{2}\alpha^2 W_0^2 w'\Sigma w}$$

$$\frac{d}{dw} : \left[-\alpha W_0 E\left(R_{t+1}^e\right) + \alpha^2 W_0^2 w'\Sigma\right]e^{(\cdot)} = 0$$

$$w = \frac{1}{\alpha W_0}\Sigma^{-1}E\left(R_{t+1}^e\right)$$

Now we have mean and variance, even closer to the "real" formula.

3)

a)

$$1 = E_t(M_{t+1}R_{t+1}) = E_t e^{-y_t^{(1)} - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 - \lambda\varepsilon_{t+1}}e^{r_{t+1}}$$

$$0 = -y_t^{(1)} - \frac{1}{2}\lambda^2\sigma_\varepsilon^2 + E_t r_{t+1} + \frac{1}{2}\sigma^2(r_{t+1} - \lambda\varepsilon_{t+1})$$

$$0 = -y_t^{(1)} + E_t r_{t+1} + \frac{1}{2}\sigma^2(r_{t+1}) - \lambda cov(r_{t+1}, \varepsilon_{t+1})$$

$$E_t r_{t+1} + \frac{1}{2}\sigma^2(r_{t+1}) = y_t^{(1)} + \lambda cov(r_{t+1}, \varepsilon_{t+1})$$

14

This is close to our friend. However, note the $1/2\sigma^2$ term. Also λ is not the risk aversion coefficient. Substituting

$$E_t r_{t+1} + \frac{1}{2}\sigma^2(r_{t+1}) = y_t^{(1)} + \gamma cov(r_{t+1}, (E_{t+1} - E_t)\Delta c_{t+1})$$

λ is the *market price of interest rate risk*, meaning how much an expected return must rise per unit of covariance with a shock to interest rates. λ is not the risk aversion coefficient.

b)

$$
\begin{aligned}
r_{t+1}^{(n)} &= p_{t+1}^{(n-1)} - p_t^{(n)} = \delta + A_{n-1} - B_{n-1}\left(y_{t+1}^{(1)} - \delta\right) - A_n + B_n\left(y_t - \delta\right) \\
&= \delta - (A_n - A_{n-1}) + [B_n - \phi B_{n-1}]\left(y_t^{(1)} - \delta\right) - B_{n-1}\varepsilon_{t+1} \\
r_{t+1}^{(n)} &= y_t^{(1)} - \left[\frac{1}{2}B_{n-1}^2 + B_{n-1}\lambda\right]\sigma_\varepsilon^2 - B_{n-1}\varepsilon_{t+1}
\end{aligned}
$$

where $B_{n-1} = \left(\frac{1-\phi^{n-1}}{1-\phi}\right)$. The final term is negative: interest rate rises make long term bonds go down. (Duh, but worth checking the sign.)

c) Check:

$$
\begin{aligned}
E_t r_{t+1}^{(n)} &= y_t^{(1)} - \frac{1}{2}B_{n-1}^2\sigma_\varepsilon^2 - B_{n-1}\lambda\sigma_\varepsilon^2 \\
\sigma^2\left(r_{t+1}^{(n)}\right) &= B_{n-1}^2\sigma_\varepsilon^2 \\
cov(r_{t+1}^{(n)}, \varepsilon_{t+1}) &= -B_{n-1}\sigma_\varepsilon^2
\end{aligned}
$$

d) The return risk premium is

$$E_t r_{t+1}^{(n)} - y_t^{(1)} = -\left[\frac{1}{2}B_{n-1}^2 + B_{n-1}\lambda\right]\sigma_\varepsilon^2$$

The forward rate risk premium is

$$f_t^{(n)} - E_t y_{t+n-1}^{(1)} = \delta - \left[\frac{1}{2}B_{n-1}^2 + B_{n-1}\lambda\right]\sigma_\varepsilon^2 + \phi^{n-1}\left(y_t^{(1)} - \delta\right) - \left[\delta + \phi^{n-1}\left(y_t^{(1)} - \delta\right)\right]$$

$$f_t^{(n)} - E_t y_{t+n-1}^{(1)} = -\left[\frac{1}{2}B_{n-1}^2 + B_{n-1}\lambda\right]\sigma_\varepsilon^2$$

They are the same. This may remind you of the Fama-Bliss result, but be careful, it is something totally different. We are looking at a constant risk premium, where FB were looking at the variation of the risk premium through time. Everything we are doing here gets soaked up into the constant in the FB regression. In both cases $\lambda > 0$ implies a negative risk premium – expected returns are lower for long term bonds, and forward rates are lower than expected future spot rates.

e)

$$
\begin{aligned}
\Delta c_{t+1} &= a + x_t + v_{t+1} \\
\Delta c_{t+1} &= a + (1 - \phi L)^{-1}\frac{1}{\lambda}v_t + v_{t+1} \\
(1 - \phi L)\Delta c_{t+1} &= a(1 - \phi) + \frac{1}{\lambda}v_t + (1 - \phi L)v_{t+1} \\
(1 - \phi L)\Delta c_{t+1} &= a(1 - \phi) + v_{t+1} + (\frac{1}{\lambda} - \phi)v_t \\
(1 - \phi L)\Delta c_{t+1} &= a(1 - \phi) + (1 - \theta L)v_{t+1} \\
\theta &= \phi - \frac{1}{\lambda}
\end{aligned}
$$

Thus, Δc_{t+1} follows an ARMA(1,1) with roots $-\phi$ and $\theta = \frac{1}{\lambda} - \phi$. This is an important lesson *models of the form (??) have ARMA(1,1) univariate representations.* Also this trick for finding the univariate representation is useful. In *Asset Pricing* I needlessly wasted pages matching spectral densities.

f) Iid happens when $\lambda \to \infty$. Then $\sigma^2(x) \to 0$, expected returns do not vary over time. In the original model, $\sigma_v^2 = $ constant and $\lambda \to \infty$ means

$$v_{t+1} = \frac{\lambda}{\gamma}\varepsilon_{t+1},$$

$$\varepsilon_{t+1} = \frac{\gamma}{\lambda}v_{t+1}$$

$$\sigma_\varepsilon^2 = \frac{\gamma^2}{\lambda^2}\sigma_v^2$$

$$\lambda\sigma_\varepsilon^2 = \frac{\gamma^2}{\lambda}\sigma_v^2$$

Thus, all the $\lambda\sigma_\varepsilon^2$ terms go smoothly to zero. *Risk premiums are zero in this case.*

The original parameterization was not great to see this, because it held constant the variance of interest rate innovations. If consumption growth becomes iid, there are no interest rate innovations. I hope reparameterizing and translating was easier for you to see than taking this funky limit of the original model.

g)

$$(1 - \phi L)\Delta c_{t+1} = a + (1 - \theta L)v_{t+1}$$

$$\theta = \phi - \frac{1}{\lambda}$$

$$
\begin{aligned}
\Delta c_{t+1} &= \frac{a}{1-\phi} + \frac{1 - \theta L}{1 - \phi L}v_{t+1} \\
&= \frac{a}{1-\phi} + \left(1 + \frac{(\phi - \theta)L}{1 - \phi L}\right)v_{t+1} \\
&= \frac{a}{1-\phi} + (1 + (\phi - \theta)\left[L + \phi L^2 + \phi^2 L^3\right] + ..)v_{t+1}
\end{aligned}
$$

Thus, if $\lambda > 0$, $\theta < \phi$, and the impulse-response function is positive throughout. It has a big spike at 0, and then a slow decay at rate ϕ. The ARMA(1,1) captures the spike – the possibility for a big unexpected return that does not move expected returns as dramatically as an AR(1).

ARMA(1,1) impulse response