Problem Set 11 Answers

1.

(a) From lecture,

$$\max_{[w]} E\left[u(c_{t+1})\right]$$

$$\begin{aligned} c_{t+1} &= W_{t+1} = R^p_{t+1} W_t \\ R^p_{t+1} &= (1-w) R^f_t + w R_{t+1} = R^f_t + w R^e_{t+1} \end{aligned}$$

$$\max_{[w]} E\left\{ u\left[\left(R^f_t + w R^e_{t+1} \right) W_t \right] \right\}$$

$$\frac{d}{dw} : E\left\{ u'\left[\left(R^f_t + w R^e_{t+1} \right) W_t \right] R^e_{t+1} \right\} = 0$$

(b) Quadratic.

$$\begin{aligned} u(c) &= -\frac{1}{2}\left(c^* - c\right)^2 \\ u'(c) &= c^* - c \end{aligned}$$

$$\begin{aligned} E\left\{ \left(c^* - \left(R^f_t + w R^e_{t+1} \right) W_t \right) R^e_{t+1} \right\} &= 0 \\ E\left\{ \left(c^* - R^f_t W_t \right) \left[E\left(R^e_{t+1} \right) - w W_t E\left(R^{e2}_{t+1} \right) \right] \right\} &= 0 \end{aligned}$$

$$w = \frac{\left(c^* - R^f_t W_t \right)}{W_t} \frac{E(R^e_{t+1})}{\left[E\left(R^e_{t+1} \right)^2 + \sigma^2(R^e_{t+1}) \right]}$$

That's a good enough answer, but there is a prettier way to express it. Note that relative risk aversion is

$$\gamma = \frac{-cu''(c)}{u'(c)} = \frac{c}{(c^* - c)}$$

So, we can express the answer as

$$w = \frac{1}{\gamma} R^f \frac{E(R^e_{t+1})}{\left[E\left(R^e_{t+1} \right)^2 + \sigma^2(R^e_{t+1}) \right]}$$

if we define

$$\gamma = \frac{-cu''(c)}{u'(c)} = \frac{R^f W_t}{(c^* - R^f W_t)}$$

as the local coefficient of risk aversion, evaluated at the point $c_{t+1} = R^f W_t$ that would be generated by putting it all in the risk free rate. So, investors who are less risk averse invest more in stocks. This is not exactly the

$$w = \frac{1}{\gamma} \Sigma^{-1} \mu$$

formula that holds in continuous time, but it's pretty close, no?

(c) Example 2: normal-exponential.

$$u(c) = -e^{-\alpha c}$$

$$E\left[-e^{-\alpha\left(R_t^f + wR_{t+1}^e\right)W_t}\right] = -e^{-\alpha R_t^f W_t - \alpha w E\left(R_{t+1}^e\right)W_t + \frac{1}{2}\alpha^2 w^2 W_t^2 \sigma^2(R_{t+1}^e)}$$

$$\frac{d}{dw} : \left(-\alpha E\left(R_{t+1}^e\right) W_t + \alpha^2 w W_t^2 \sigma^2(R_{t+1}^e)\right) e^{(\cdot)} = 0$$

$$-\alpha E\left(R_{t+1}^e\right) + \alpha^2 w W_t \sigma^2(R_{t+1}^e) = 0$$

$$w = \frac{1}{\alpha W_t} \frac{E\left(R_{t+1}^e\right)}{\sigma^2(R_{t+1}^e)}$$

Now we have mean and variance, even closer to the "real" formula. (αW is the coefficient of relative risk aversion.)

2.

(a)

$$\frac{W_T^{1-\gamma}}{W_0^{1-\gamma}} = e^{(1-\gamma)\left[w\left(\mu - r^f\right) + r^f - \frac{1}{2}w^2\sigma^2\right]T + (1-\gamma)w\sigma\sqrt{T}\varepsilon}$$

$$\begin{aligned}
\frac{E\left[W_T^{1-\gamma}\right]}{W_0^{1-\gamma}} &= e^{(1-\gamma)\left[r^f + w\left(\mu - r^f\right) - \frac{1}{2}w^2\sigma^2\right]T + \frac{1}{2}(1-\gamma)^2 w^2\sigma^2 T} \\
&= e^{(1-\gamma)\left[r^f + w\left(\mu - r^f\right) - \frac{1}{2}w^2\sigma^2 + \frac{1}{2}(1-\gamma)w^2\sigma^2\right]T} \\
&= e^{(1-\gamma)\left[r^f + w\left(\mu - r^f\right) - \frac{1}{2}\gamma w^2\sigma^2\right]T}
\end{aligned}$$

(b)

$$\frac{d}{dw}\left\{e^{(1-\gamma)\left[r^f + w\left(\mu - r^f\right) - \frac{1}{2}\gamma w^2\sigma^2\right]T}\right\} = (1-\gamma)\left[\left(\mu - r^f\right) - \gamma w\sigma^2\right] T e^{(1-\gamma)\left[r^f + w\left(\mu - r^f\right) - \frac{1}{2}\gamma w^2\sigma^2\right]T} = 0$$

$$\left(\mu - r^f\right) - \gamma w\sigma^2 = 0$$

$$w = \frac{\left(\mu - r^f\right)}{\gamma\sigma^2}$$

Hooray! Our most basic formula.

(c) T drops from the formulas. *In an iid lognormal world with power utility, the optimal allocation is independent of the investment horizon*

(d) Calculation is right, implication is not.

$$\begin{aligned}
\sigma^2\left[\frac{1}{T}(r_1 + r_2 + ... + r_T)\right] &= \frac{1}{T^2}T\sigma^2(r) = \frac{1}{T}\sigma^2(r) \\
E(r_1 + r_2 + ... + r_T) &= TE(r) \\
\sigma^2(r_1 + r_2 + ... + r_T) &= T\sigma^2(r) \\
\frac{E(r_1 + r_2 + ... + r_T)}{\sigma(r_1 + r_2 + ... + r_T)} &= \sqrt{T}\frac{E(r)}{\sigma(r)}
\end{aligned}$$

The variance of average (annualized) returns does decrease, and Sharpe ratios do increase. But portfolio theory wants the *variance* of the *total* return. Even in these simple calculations

$$\frac{E(r_1 + r_2 + ... + r_T)}{\sigma^2(r_1 + r_2 + ... + r_T)} = \frac{E(r)}{\sigma(r)}$$

independent of horizon.

(e) In reality, returns are predictable ("stocks are a bit like long term bonds") and covary with their state variable - when dp goes down, expected returns go down, but price and return go up, hedging the reinvestment risk. Also, people may have outside income streams.

3.

(a) The key here is to separate the "active" and "passive" portfolios.

$$w^m = \frac{1}{\gamma}\frac{E(R^{em})}{\sigma^2(R^{em})} = \frac{1}{2}\frac{0.08}{0.20^2} = \frac{1}{2}\frac{0.08}{0.04} = 1.$$

The resdiual covariance matrix, with $\sigma(\varepsilon) = 0.10$ and thus $\sigma^2(\varepsilon) = 0.01$ and $\rho = -0.5$ is

$$\Sigma = 0.01\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = 0.01\begin{bmatrix} 1 & -0.5 \\ 0.5 & 1 \end{bmatrix}.$$

and

$$\Sigma^{-1} = \frac{1}{0.01}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1} = \frac{1}{0.01}\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}.$$

Hence, the weights on the alpha part are

$$\begin{aligned}
w_\alpha &= \frac{1}{\gamma}\Sigma^{-1}\alpha = \frac{1}{2}\frac{1}{0.01}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\
&= \frac{1}{2}\frac{1}{0.01}\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\
&= \frac{1}{2}\frac{1}{0.01}\frac{1}{0.75}\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} -0.003 \\ 0.012 \end{bmatrix} \\
&= \frac{1}{2}\frac{4}{3}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} -0.3 \\ 1.2 \end{bmatrix} \\
&= \frac{1}{2}\frac{4}{3}\begin{bmatrix} 0.3 \\ 1.05 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.7 \end{bmatrix}
\end{aligned}$$

This expresses how much you should invest in the market excess returns and beta-hedged portfolios, $\alpha^i + \varepsilon^i = R^{ei} - \beta_i R^{em}$. I find it useful to write the portfolio as

$$\begin{aligned}
R^{ep} &= 1.0 \times R^{em} + w_1 \times \left(R^{e1} - \beta_1 R^{em}\right) + w_2 \times \left(R^{e2} - \beta_2 R^{em}\right) \\
R^{ep} &= 1.0 \times R^{em} + 0.2 \times \left(R^{e1} - \beta_1 R^{em}\right) + 0.7 \times \left(R^{e2} - \beta_2 R^{em}\right)
\end{aligned}$$

(b) In terms of the actual excess returns, we have to pull out the betas

$$\begin{aligned}
R^{ep} &= 1.0 \times R^{em} + 0.2 \times \left(R^{e1} - 2R^{em}\right) + 0.7 \times \left(R^{e2} - 2R^{em}\right) \\
R^{ep} &= (1.0 - 0.4 - 1.4) \times R^{em} + 0.2 \times R^{e1} + 0.7 \times R^{e2} \\
R^{ep} &= -0.8 \times R^{em} + 0.2 \times R^{e1} + 0.7 \times R^{e2}
\end{aligned}$$

The manager weights don't change, but the market weight now reflects the beta offset that was originally bundled. *It's a big deal.* Get the passive portfolio beta right! *Offset the beta exposures of your active managers.* In this case these active guys have so much beta that you are actually short the market to offset their beta exposure!

(c) In terms of returns, we now pull R^f out just like we pulled R^{em} out in the last one.

$$\left(R^p - R^f\right) = -0.8 \times \left(R^m - R^f\right) + 0.2 \times \left(R^1 - R^f\right) + 0.7 \times \left(R^2 - R^f\right)$$

$$R^p = (1 + 0.8 - 0.2 - 0.7)\, R^f - 0.8 \times R^m + 0.2 \times R^1 + 0.7 \times R^2$$

$$R^p = 0.9R^f - 0.8 \times R^m + 0.2 \times R^1 + 0.7 \times R^2$$

(d) I hope you see in this problem that the clever way I set it up in a) gives useful numbers. The weights on the managers are always the same. However, the weights on the market return and risk free rate then adjust a lot according to how we express the problem. Pulling the beta hedges out of the manager's portfolio turns a passive long position in the market into a passive short position! I think the first version is the most intuitive, but a big point is to understand how it's different from "where you really put your money."

(e) Yes, you invest positively in a negative alpha manager! The reason is his negative correlation – the portfolio of the two managers exploits the good manager's alpha and uses the bad manager's ε to diversify. If the ε were uncorrelated, then

$$\Sigma^{-1}\alpha = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1/\sigma_1^2 \\ \alpha_2/\sigma_2^2 \end{bmatrix}$$

and positive α must correspond to positive weight.

4.

(a)

$$w = \frac{1}{2}\frac{E(R^e)}{\sigma^2(R^e)} = \frac{1}{2}\frac{0.08}{0.20^2} = \frac{1}{2}\frac{0.08}{0.04} = 1$$

(b)

$$\begin{aligned} w &= \frac{1}{2}\frac{E(R^e)}{\sigma^2(R^e)+\sigma^2(E(R^e))} = \frac{1}{2}\frac{0.08}{0.20^2+0.05^2} = \frac{1}{2}\frac{0.08}{0.04+0.05^2} \\ &= \frac{0.04}{0.04+0.0025} = \frac{0.04}{0.04+0.0025} = \frac{0.04}{0.0425} \approx 0.94 \end{aligned}$$

(c)

$$\begin{aligned} w &= \frac{1}{2}\frac{E(R^e)}{\sigma^2(R^e)+\sigma^2(E(R^e))} = \frac{1}{2}\frac{0.08\times 10}{0.20^2\times 10+(0.05\times 10)^2} = \frac{0.04}{0.04+0.05^2\times 10} \\ &= \frac{0.04}{0.04+0.0025\times 10} = \frac{0.04}{0.04+0.025} = \frac{0.04}{0.0625} \approx 0.64 \end{aligned}$$

(d) It's a big difference. Notice the reason why – *variance* grows linearly with horizon, while *standard error* grows with the square of horizon.

5. Yes! It would be a great portfolio for small business owners to *short* in order to hedge their risks! So you should set up your fund to constantly maintain a short position in this strategy.

6. The formula

$$w = \frac{1}{\gamma}\frac{E_t(R^e)}{\sigma^2(R^e)} + \frac{\eta}{\gamma}\beta_{R^e,y}$$

Since the covariance is zero, the hedging demand is zero. However, it would be a great market-timing signal: invest more if the NL wins the world series. Point: hedging demand depends on the covariance of signal and returns, as bond returns are negatively correlated with bond yields.

7. The formula

$$w = \frac{1}{\gamma}\frac{E_t(R^e)}{\sigma^2(R^e)} + \frac{\eta}{\gamma}\beta_{R^e,y}$$

The first term is zero. However, yield is the "state variable" for long term bonds. And bond returns are strongly correlated with yield shocks. Hence an investor with a big η will want to hold long term bonds for their hedging purpose. Which is just a fancy way of saying an investor with a 10 year horizon holds 10 year bonds.