Business 35904 **John H. Cochrane**

Problem Set 1

Due Monday, October 21

1. Do problem 2 in Ch 2 *Asset Pricing* Don't worry about the technicalities needed to get to the present value form of the resource constraint,

$$k_t + \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j} = \sum_{j=0}^{\infty} \beta^{j+1} E_t c_{t+j}.$$

Yes, a "transversality condition" $\lim_{T\to\infty} \beta^T k_{t+T} = 0$ is needed, and yes, there is some danger of the consumer just sitting at his bliss point c^*. Treat consumption as *not* freely disposable and otherwise ignore these problems.

Hint: Find the first order condition and show that consumption follows a random walk. Then figure out what the innovation of the random walk has to be to satisfy the above present value resource constraint. Here is where you use general equilibrium and the production technology. To get to (2.7) just take innovations of (2.6) without doing lots of algebra.

2. Repeat problem 2 in continuous time. The problem is

$$\begin{aligned} & \max -\frac{1}{2} E \int e^{-rt} \left(c^* - c_t\right)^2 \text{ s.t.} \\ dk_t &= \left(rk_t + y_t - c_t\right) dt \\ dy_t &= -\rho y_t dt + \sigma dz_t \end{aligned}$$

(I use y rather than e for the endowment so we don't mix it up with e^{rt}). You follow the same steps

(a) Express the resource constraint in "present value" form;

(b) Find the first order conditions relating c_t and c_{t+k}

(c) Specialize to c_t and $c_{t+\Delta}$, and thus express the first-order condition as a restriction on dc_t; (the "random walk") in continuous time.

(d) Plug from the first order condition into the "present value" resource constraint to obtain the continuous-time analogues to (2.6) and (2.7). (You will find (2.7) from (2.6) by taking its d, i.e. from $c_t = ...$you can find $dc_t = rdk_t + ...$ You can't "take innovations" as we just apply $E_{t+1} - E_t$ in discrete time. The operation $E_{t+\Delta} - E_t$ really doesn't exist in continuous time. Instead, we take $dx_t = \mu dt + \sigma dz_t$ and identify σdz_t as the innovation.)

(e) Argue that the risk-free rate must equal the technological rate of transformation, $r_t^f = r$. Hint: $r_t^f = -E_t\left(\frac{d\Lambda_t}{\Lambda_t}\right)$

(f) Find the price of the consumption stream from

$$p_t = E_t \int_{s=0}^{\infty} \frac{\Lambda_{t+s}}{\Lambda_t} c_{t+s} ds$$

From step d you will have a consumption process that will let you evaluate the required $E_t c_{t+j}^2$ terms.

(g) (You can see by the stock price that there will be time varying risk premiums in this model. I wanted to have you characterize $E_t(dR_{t+1}) = f(c_t - c^*)$.It's straightforward but not algebraically pretty. But look enough to see that you could do it if you had to)

This looks daunting but it is actually much easier in continuous time than discrete. But you have to think straight about how it works!

Big comment: This problem is a general equilibrium model in which the riskfree technology nails down one asset price, the riskfree rate, but technology does not nail down risk premiums, as there is no technological way to replicate risky assets. So it lies halfway on the general equilibrium spectrum between endowment economies, like the next problem, and economies with risky linear technologies that nail down all asset returns, like the standard finance paradigm. In those economies, quantities are completely endogenous. You got some sense of that here in that much of the consumption stream had to be derived.

3. (This is an improved version of problem 3, Ch2. It is the Mehra-Prescott model which launched the equity premium, and it shows you how to think about a pure endowment economy. It's also a good introduction/reminder of recursive tools.). In this problem you will construct a complete simple simulation economy.

   The economy has either good or bad states; consumption either grows by $u = 1.06$ (i.e. 6%) or goes down by $d = 0.98$ (down 2%). If the economy growing in the u state, it has probability $\pi_{u\to u} = (1+x)/2$ of remaining in the good state, and probability $\pi_{u\to d} = (1-x)/2$ of going to the bad state. x is a parameter which we'll specify later. If the economy is in the bad state d it has probability $\pi_{d\to u} = (1-x)/2$ of going to the good state, and probability $\pi_{d\to d} = (1+x)/2$ of remaining in the bad state. As you can see, the parameter x controls the persistence of the state of the economy. When $x = 0$, consumption growth is independent over time like a coin flip, all probabilities are 1/2. As x increases, once you are in a good or bad state, you are more and more likely to stay there. Growth is thus autocorrelated. (It would be more interesting to have a slow-moving state variable to describe growth, but we'll do that some other time.)

   Investors have power utility

   $$E\sum_{j=0}^{\infty}\beta^j \frac{c_{t+j}^{1-\gamma}}{1-\gamma}$$

   with $\beta = 0.99$

   I'll show you how to find the price of a perpetuity, i.e. a bond that pays a coupon of \$1 per period forever. You will use the same ideas to find the one period interest rate and the stock price. The value of a perpetuity at any date is

   $$p_t^p = E_t \sum_{j=1}^{\infty}\beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} \times 1$$

   A direct attack leads to a mountain of algebra. Instead, think about the one period relation

   $$p_t^p = E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(1+p_{t+1}^p\right)\right].$$

   All information about the future in this economy is contained in the current value of consumption growth. (Consumption growth is a *state variable* for this economy.) Thus p_t^p and p_{t+1}^p can *only* depend on consumption growth at t and $t+1$ respectively. p^p can only take on two values, $p^p(u)$ in the good state and $p^p(d)$ in the bad state. We can find these prices as follows

   $$\begin{cases} p^p(u) = \beta\pi_{u\to u}u^{-\gamma}\left[1+p^p(u)\right] + \beta\pi_{u\to d}d^{-\gamma}\left[1+p^p(d)\right] \\ p^p(d) = \beta\pi_{d\to u}u^{-\gamma}\left[1+p^p(u)\right] + \beta\pi_{d\to d}d^{-\gamma}\left[1+p^p(d)\right] \end{cases}$$

   This is a system of equations in the two unknowns $p^p(u)$ and $p^p(d)$. One fun way to solve this is to start with $p^p(u) = 0$ and $p^p(d) = 0$ on the right hand side, plug through to find new values on the left hand side, use those new values on the right hand side, and so on. This will converge, because you're finding at each step j the value of a series of coupons that lasts j periods. This approach is useful in more complicated and nonlinear model. Here though, the system is linear and it's easy enough to just solve for $p^p(u)$ and $p^p(d)$.

$$\begin{cases} \left(1-\beta\pi_{u\to u}u^{-\gamma}\right)p^p(u)-\beta\pi_{u\to d}d^{-\gamma}p^p(d)=\beta\pi_{u\to u}u^{-\gamma}+\beta\pi_{u\to d}d^{-\gamma} \\ \left(1-\beta\pi_{d\to d}d^{-\gamma}\right)p^p(d)-\beta\pi_{d\to u}u^{-\gamma}p^p(u)=\beta\pi_{d\to u}u^{-\gamma}+\beta\pi_{d\to d}d^{-\gamma} \end{cases}$$

$$\begin{bmatrix} 1-\beta\pi_{u\to u}u^{-\gamma} & -\beta\pi_{u\to d}d^{-\gamma} \\ -\beta\pi_{d\to u}u^{-\gamma} & 1-\beta\pi_{d\to d}d^{-\gamma} \end{bmatrix}\begin{bmatrix} p^p(u) \\ p^p(d) \end{bmatrix}=\begin{bmatrix} \beta\pi_{u\to u}u^{-\gamma}+\beta\pi_{u\to d}d^{-\gamma} \\ \beta\pi_{d\to u}u^{-\gamma}+\beta\pi_{d\to d}d^{-\gamma} \end{bmatrix}$$

$$\begin{bmatrix} p^p(u) \\ p^p(d) \end{bmatrix}=\begin{bmatrix} 1-\beta\pi_{u\to u}u^{-\gamma} & -\beta\pi_{u\to d}d^{-\gamma} \\ -\beta\pi_{d\to u}u^{-\gamma} & 1-\beta\pi_{d\to d}d^{-\gamma} \end{bmatrix}^{-1}\begin{bmatrix} \beta\pi_{u\to u}u^{-\gamma}+\beta\pi_{u\to d}d^{-\gamma} \\ \beta\pi_{d\to u}u^{-\gamma}+\beta\pi_{d\to d}d^{-\gamma} \end{bmatrix}.$$

I couldn't make it look pretty by expanding the right hand side, but this is good enough to plug in numbers and find the price in the two states.

Your objective is to write a little program that will find the numbers, such as $p^p(u)$ and $p^p(d)$ for given values of the parameters. You do *not* have to find pretty algebraic expressions. Start your program with $\gamma=1$ and $x=0$ (but leave these parameters as variables so we can change them later.)

(a) Now, it's your turn. Find the one period interest rate in this economy. Like everything else, the interest rate can take on only two values, one value in the good state and a different value in the bad state. In each state, the interest rate is

$$R_t^f=1/E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]$$

Thus, your job is to find the formula for the interest rate $R^f(u), R^f(d)$.

(b) To model a stock, we can think of an asset that pays consumption as its dividend. The value of the whole U.S. economy is the value of a claim to the consumption it can provide us. Now, the stock *price*

$$p_t=E_t\sum_{j=1}^{\infty}\beta^j\left(\frac{c_{t+j}}{c_t}\right)^{-\gamma}c_{t+j}$$

depends on the *level* of consumption, not just the growth rate at each date. However, we can apply the same trick as for the perpetuity to the ratio of price to consumption, i.e. the price/dividend ratio of the stock,

$$\begin{aligned} p_t &= E_t\left[m_{t+1}\left(p_{t+1}+c_{t+1}\right)\right] \\ \frac{p_t}{c_t} &= E_t\left[m_{t+1}\left(\frac{p_{t+1}}{c_{t+1}}+1\right)\frac{c_{t+1}}{c_t}\right] \\ \frac{p_t}{c_t} &= E_t\left[\beta\left(\frac{p_{t+1}}{c_{t+1}}+1\right)\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}\right] \end{aligned}$$

$\frac{p}{c}$ can take on only two values, $\frac{p}{c}(u)$ and $\frac{p}{c}(d)$. Proceed as with the perpetuity to find a formula for those two values.

(c) Find a formula for stock returns and excess returns (return- riskfree rate), and the long term bond returns and excess returns. For each one, you have to find *four* values, the return *from* each state u and d *to* states u and d. Hint: for the stock return:

$$R_{t+1}=\frac{p_{t+1}+c_{t+1}}{p_t}=\frac{\left(\frac{p_{t+1}}{c_{t+1}}+1\right)\frac{c_{t+1}}{c_t}}{\frac{p_t}{c_t}}$$

(d) Find a formula for the *expected* stock and perpetuity returns in the two states. If something takes two values $x(u), x(d)$, then its expected value if today is u is $E_t(x) = \pi_{u\to u}x(u) + \pi_{ud}x(d)$.

(e) Find numerical values for these quantities for the following parameter configurations:

i. $\gamma = 1$, $x = 0$

ii. $\gamma = 1$, $x = 0.3$. Motivation: everything looks the same across states in i. If the world is iid, p/d is constant. Let's add some autocorrelation in consumption growth. Results: Are interest rates higher in the u state or the d state? (Important: interest rates depend on what's expected to happen next period, not what's happening today!) Are bond prices higher or lower in the u state? Why? Are stock prices (p/c) higher or lower in the good state? Why?

iii. $\gamma = 5$, $x = 0.3$ Motivation: The equity premium is tiny. Maybe raising the risk aversion coefficient will give us a more reasonable expected stock excess returns. Now, are stock prices (p/c) greater in the good state or the bad state? Why? Is the equity premium positive or negative? Why? (This is a great problem for illustrating that general equilibrium gives unanticipated results and endogenous betas!)

iv. $\gamma = 5$, $x = -0.3$ Motivation: iii gives you a *negative* equity premium! Maybe reversing x will fix that.

For each set of parameters, make a table with the bond price, the stock price/consumption ratio, the riskfree rate, the expected excess stock and bond returns, and the actual stock and bond returns *to* the two following states. For example, here is a table of my results for the first question. (I report returns as percentages, $r = (R - 1) \times 100$.) *Explain the behavior of the variables.* Why are p or p/c higher or lower in one state than the other? Why is the equity or bond expected excess return positive or negative? Use the economics of the class and readings, for example starting with the intuition behind $R^f \approx 1 + \delta + \gamma E_t(\Delta c_{t+1})$ to explain interest rate behavior. (You may find it helpful to print out extra variables, like consumption growth, expected consumption growth, returns rather than just excess returns, and so forth. You may also find it useful to simulate some data and plot the variables over time. However, this is not required.) What we want here is to find parameters that generate prices and returns like the ones we see in the data – a decent equity premium, slow-moving price/dividend ratios that forecast returns, and so forth. We won't get there, but we'll learn how hard it is!

My program fills in the following table:

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Results for      u       d    beta  gamma       x     phi
              1.06    0.98    0.99   1.00    0.00    0.00

                Dc    E dC Bond p     p/c      Rf      ER     ERp  ER-Rf ERp-Rf
good state u  6.00    2.00  34.82 .....
bad  state d  .....

              Stock Return  stock excess   Bond Return   bond excess
 To state:       u       d       u       d       u       d       u       d
   From u:    7.07  .....
   From d:    .....
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