Problem set 1 Answers:

1.

(a) The first order conditions are

$$u'(c_t) = E_t\left[R\beta u'(c_{t+1})\right]$$
$$c_t - c^* = E_t\left[R\beta(c_{t+1} - c^*)\right]$$

with $R = 1 + r$, $R\beta = 1$ so

$$c_t = E_t\left(c_{t+1}\right).$$

Consumption follows a random walk. This is approximately true in many nonlinear models. Now we know

$$c_{t+1} = c_t + v_{t+1}$$

for some shock v_{t+1} The rest of the job – and the content of this "general equilibrium model" – is to figure out how to tie v_{t+1} to technological innovations, i.e. to impose the "budget constraint."

The first thing we do is express the technology as a "present value budget constraint."

$$k_t + \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j} = \sum_{j=0}^{\infty} \beta^{j+1} E_t c_{t+j}.$$

Intuitively, the present value of future consumption must equal wealth plus the present value of future endowment (labor income). To get here, iterate the technology forward,

$$k_{t+2} = R\left(Rk_t + i_t\right) + i_{t+1} = R^2 k_t + Ri_t + i_{t+1}$$

$$\begin{aligned}
k_{t+3} &= R^3 k_t + R^2 i_t + Ri_{t+1} + i_{t+2} \\
\frac{1}{R^3} k_{t+3} &= k_t + \frac{1}{R}\left[i_t + \frac{1}{R} i_{t+1} + \frac{1}{R^2} i_{t+2}\right] \\
\beta^3 k_{t+3} &= k_t + \beta\left[i_t + \beta i_{t+1} + \beta^2 i_{t+2}\right]
\end{aligned}$$

Continuing and with the transversality condition $\lim_{T\to\infty} \beta^T k_{t+T} = 0$, and $i_t = e_t - c_t$

$$k_t + \sum_{j=0}^{\infty} \beta^{j+1} e_{t+j} = \sum_{j=0}^{\infty} \beta^{j+1} c_{t+j}$$

This holds *ex-post*. We can take expectations using *any* information set,

$$k_t + \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j} = \sum_{j=0}^{\infty} \beta^{j+1} E_t c_{t+j}.$$

Intuitively, the present value of future consumption must equal wealth plus the present value of future endowment (labor income). The $j + 1$ comes from the timing, alas standard in the macro literature and national income accounts. If you adopt the more common finance timing convention

$$k_{t+1} = (1 + r)\left(k_t + i_t\right)$$

you get more natural present value formulas with β^j rather than β^{j+1}.

Now, substitute the first order condition in the "budget constraint" ("resource constraint," "technology" or "production possibility frontier" if you want the General Equilibrium interpretation)

$$\begin{aligned} k_t + \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j} &= \sum_{j=0}^{\infty} \beta^{j+1} c_t = \\ \beta \frac{1}{(1-\beta)} c_t &= \frac{1}{R} \frac{1}{(1-\frac{1}{R})} c_t = \frac{1}{R-1} c_t = \frac{c_t}{r} \end{aligned}$$

$$c_t = r k_t + r \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j}.$$

Consumption equals the annuity value of wealth (capital) rk_t plus the present value of future labor income (endowment).

This is the *permanent income hypothesis.* It is not necessarily a "partial equilibrium" result about a consumer facing income and interest rates, resulting in a "consumption function" to be added to some other model. It can be interpreted as a full general equilibrium model with linear technology and an endowment income process. (Any old-fashioned "partial equilibrium" result can easily be dressed up as "general equilibrium" by saying "linear technology" instead of "price," "wage" or "interest rate.")

Now to the random walk in consumption. Take innovations of the PIH,

$$(c_t - E_{t-1} c_t) = r (k_t - E_{t-1} k_t) + r \sum_{j=0}^{\infty} \beta^{j+1} (E_t - E_{t-1}) e_{t+j}.$$

the first term is zero,

$$\begin{aligned} k_t &= (1+r) k_{t-1} + e_{t-1} - c_{t-1} \\ (k_t - E_{t-1} k_t) &= 0 \end{aligned}$$

and we already had

$$c_{t-1} = E_{t-1} c_t$$

thus,

$$c_t - c_{t-1} = r \sum_{j=0}^{\infty} \beta^{j+1} (E_t - E_{t-1}) e_{t+j}.$$

Consumption is a *random walk.* We knew that from the first order conditions, $c_t = E_t c_{t+1}$. With a full equilibrium model we can now relate the innovations to consumption to fundamental shocks to technology. In this model, changes in consumption equal the *innovation* in the present value of future income.

Bob Hall (1979) noticed the random walk nature of consumption in this model, and suggested testing it by running regressions of Δc_t on any variable at time $t-1$. This paper was a watershed. It is the first "Euler equation" test of a model; note it does not *require* the full model solution tying the shocks in Δc_t to fundamental taste and technology shocks – the second term in our random walk equation. The Hansen-Singleton (1982) Euler equation tests generalize to non-quadratic utility, random asset returns for which it is impossible to fully solve the model.

From an empirical point of view, we added a lot of extra structure to get to (**??**). If income follows a different process, or if people see variables that forecast income which we cannot see, (**??**) is wrong but the random walk is still right. Most of all, we cannot just "model" income, we have to have the actual income process as seen by investors. On the other hand of course, first order conditions alone, such as we use 99% of the time in asset pricing, leave out the question "where to the real shocks come from" which we have to get to at some point.

Technical details: I have assumed no free disposal - you follow the first order conditions even if past the bliss point. If you can freely dispose of consumption, then you will always end up at the bliss point c^* sooner or later. (Thanks to Ashley Wang for pointing this out. Hansen and Sargent's treatments of this problem deal with the bliss point issue.)

By the way, the algebra is much easier if you use lag operators, i.e. write

$$c_t = rk_t + r\beta E_t\left[(1-\beta L^{-1})^{-1}e_t\right]$$

and use the Hansen-Sargent prediction formulas. But if you know how to do that, you've probably seen this model before.

(b)

$$c_t = rk_t + r\sum_{j=0}^{\infty}\beta^{j+1}E_t e_{t+j} = rk_t + r\beta\sum_{j=0}^{\infty}\beta^j\rho^j e_t = rk_t + \frac{r\beta}{1-\beta\rho}e_t.$$

$$c_t = c_{t-1} + (E_t - E_{t-1})\, r\beta\sum_{j=0}^{\infty}\beta^j e_{t+j} = c_{t-1} + r\beta\sum_{j=0}^{\infty}\beta^j\rho^j\varepsilon_t = c_{t-1} + \frac{r\beta}{1-\beta\rho}\varepsilon_t.$$

The top equation does look like a consumption function, but notice that the parameter relating consumption c to income e depends on the persistence of income e. It is not a "psychological law" or a constant of nature. If the government changes policy so that income is more unpredictable (i.e. it gets rid of the predictable part of recessions), then this coefficient declines dramatically. The income coefficient is not "policy-invariant." This is the basis of Bob Lucas' (1974) dramatic deconstruction of Keynesian models based on consumption functions that were used for policy experiments.

In both equations, you see that consumption responds to "permanent income" and that as shocks get more "permanent" – as ρ rises – consumption moves more.

(c) R was the rate of return on technology. Despite the symbol, it is not (yet) the interest rate – the equilibrium rate of return on one-period claims to consumption. That remains to be proved. The logic is, first find c, then price things from the equilibrium consumption stream. To be precise and pedantic, call the risk free rate R^f, and

$$\frac{1}{R_t^f} = E_t\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}\right) = \beta E_t\left(\frac{c_{t+1}-c^*}{c_t-c^*}\right) = \beta\left(\frac{c_t-c^*}{c_t-c^*}\right) = \beta = \frac{1}{R}$$

Now, the fun stuff. We can approach the price of the consumption stream by brute force,

$$\begin{aligned}
p_t &= E_t\sum_{j=1}^{\infty}m_{t,t+j}c_{t+j} = E_t\sum_{j=1}^{\infty}\beta^j\frac{c^*-c_{t+j}}{c^*-c_t}c_{t+j} = E_t\sum_{j=1}^{\infty}\beta^j\frac{c^*c_{t+j}-c_{t+j}^2}{c^*-c_t} \\
&= \sum_{j=1}^{\infty}\beta^j\frac{c^*c_t - E_t\left(c_{t+j}^2\right)}{c^*-c_t} = \sum_{j=1}^{\infty}\beta^j\frac{c^*c_t - c_t^2 - var_t\left(c_{t+j}\right)}{c^*-c_t}
\end{aligned}$$

$$\begin{aligned}
c_{t+1} &= c_t + \frac{r\beta}{1-\beta\rho}\varepsilon_{t+1} \\
c_{t+2} &= c_t + \frac{r\beta}{1-\beta\rho}\left(\varepsilon_{t+1}+\varepsilon_{t+2}\right) \\
c_{t+j} &= c_t + \frac{r\beta}{1-\beta\rho}\left(\varepsilon_{t+1}+..+\varepsilon_{t+j}\right)
\end{aligned}$$

$$
\begin{aligned}
E_t(c_{t+j}) &= c_t \text{ (of course)} \\
var_t(c_{t+j}) &= j\left(\frac{r\beta}{1-\beta\rho}\right)^2 \sigma_\varepsilon^2
\end{aligned}
$$

$$
\begin{aligned}
p_t &= \sum_{j=1}^{\infty} \beta^j \frac{c_t\left(c^*-c_t\right) - j\left(\frac{r\beta}{1-\beta\rho}\right)^2 \sigma_\varepsilon^2}{c^*-c_t} \\
&= \sum_{j=1}^{\infty} \beta^j \left[c_t - \frac{j\left(\frac{r\beta}{1-\beta\rho}\right)^2 \sigma_\varepsilon^2}{c^*-c_t}\right] \\
&= \left(\sum_{j=1}^{\infty} \beta^j\right) c_t - \left(\sum_{j=1}^{\infty} j\beta^j\right) \frac{\left(\frac{r\beta}{1-\beta\rho}\right)^2 \sigma_\varepsilon^2}{c^*-c_t}
\end{aligned}
$$

$$
\sum_{j=1}^{\infty} j\beta^j = \frac{\beta}{(\beta-1)^2}
$$

$$
\begin{aligned}
p_t &= \frac{\beta}{1-\beta} c_t - \frac{\beta}{(1-\beta)^2} \frac{\left(\frac{r\beta}{1-\beta\rho}\right)^2 \sigma_\varepsilon^2}{c^*-c_t} \\
&= \frac{\frac{1}{1+r}}{1-\frac{1}{1+r}} c_t - \frac{\frac{1}{1+r}}{\left(1-\frac{1}{1+r}\right)^2} \frac{\left(\frac{r\beta}{1-\beta\rho}\right)^2 \sigma_\varepsilon^2}{c^*-c_t} \\
p_t &= \frac{1}{r} c_t - \frac{\beta}{(1-\beta\rho)^2} \frac{1}{c^*-c_t} \sigma_\varepsilon^2
\end{aligned}
$$

Wow. The first term is the risk-neutral price – the value of a perpetuity paying c. (Don't forget $E_t(c_{t+j}) = c_t$) The second term is a risk correction. It lowers the price. If σ_ε^2 is high – more risk –the price is lower. If ρ is high – more persistent consumption – the price is lower.

Now, the hard term – the effect of consumption. At the bliss point, the consumer is as happy as can be, and marginal utility falls to zero. Hence, the consumer is infinitely risk averse. ($u''(c)/u'(c)$ rises to infinity). There is no consumption you can give him to compensate for risk, since he's at the bliss point. No surprise that the price goes off to $-\infty$ here. As consumption rises towards the bliss point, the consumer gets more and more risk averse (u'' is constant, u' is falling), so the price declines. Above the bliss point, the consumer values consumption negatively, so the price is higher than the risk-neutral version.

This feature – that risk aversion rises as consumption rises – is obviously not a good one. Quadratic utility is best used as a local approximation. If you use a quadratic model, find a c^* that gives a sensible risk aversion, and then make sure the model doesn't get too far away! (The CAPM is a quadratic model.

Note: the *ideas* of this model represent well how general equilibrium models work. The *solution method* does not generalize well. To solve nonlinear models you can't really do this business of finding the present value resource constraint and plugging in first-order conditions. You have to use dynamic programming or other techniques. But you're looking for the same sort of answers.

Note, suppose instead the quadratic utility investor of problem 2 lives in an endowment economy, in which the endowment is given by

$$
c_t = c_{t-1} + \frac{r\beta}{1-\beta\rho} \varepsilon_t
$$

where now the shock is simply the shock to the endowment, without any connection to an income stream (which doesn't exist). How does this change affect your pricing formulas? How bad a mistake is it to "assume an endowment economy" when in fact the true economy generates consumption from production? Answer: it would be the same.

2. The problem

$$\begin{aligned}
& \max -\frac{1}{2} E \int_{t=0}^{\infty} e^{-rt} \left(c^* - c_t\right)^2 dt \\
dk_t &= \left(rk_t + y_t - c_t\right) dt \\
dy_t &= -\rho y_t dt + \sigma dz_t
\end{aligned}$$

Iterating the constraint forward and taking expectations,

$$E_t \int_{s=0}^{\infty} e^{-rs} c_{t+s} ds = E_t \int_{s=0}^{\infty} e^{-rs} y_{t+s} ds + k_t$$

The first order condition

$$c_t = E_t c_{t+s}$$

In the limit as $k \to 0$

$$E_t dc_t = 0$$

This is the obvious "random walk" in continuous time. Plugging back in to the constraint at time t,

$$\begin{aligned}
E_t \int_{s=0}^{\infty} e^{-rs} c_{t+s} ds &= E_t \int_{s=0}^{\infty} e^{-rs} y_{t+s} ds + k_t \\
c_t &= rk_t + rE_t \int_{s=0}^{\infty} e^{-rs} y_{t+s} ds,
\end{aligned}$$

The PIH in continuous time. For the AR(1) income process,

$$\begin{aligned}
c_t &= rk_t + ry_t \int_{s=0}^{\infty} e^{-rs} e^{-\rho s} ds \\
c_t &= rk_t + \frac{r}{r+\rho} y_t
\end{aligned}$$

For the differential representation, we can't find innovations because that doesn't make sense. Instead find dc_t directly.

$$\begin{aligned}
dc_t &= rdk_t + \frac{r}{r+\rho} dy_t \\
dc_t &= r\left(rk_t + y_t - c_t\right) dt + \frac{r}{r+\rho} \left(-\rho y_t dt + \sigma dz\right) \\
&= r\left(rk_t + y_t - rk_t - \frac{r}{r+\rho} y_t\right) dt + \frac{r}{r+\rho} \left(-\rho y_t dt + \sigma dz\right) \\
dc_t &= \frac{r}{r+\rho} \sigma dz_t
\end{aligned}$$

The pricing formula

$$p_t = E_t \int_{s=0}^{\infty} e^{-rs} \frac{c^* - c_{t+s}}{c^* - c_t} c_{t+s} ds$$

$$\begin{aligned}
E_t c_{t+s} &= c_t \\
E_t c_{t+s}^2 &= c_t^2 + \left(\frac{r\sigma}{r+\rho}\right)^2 t
\end{aligned}$$

$$
\begin{aligned}
p_t &= E_t \int_{s=0}^{\infty} e^{-rs} \frac{c^* c_t - c_t^2 - \left(\frac{r\sigma}{r+\rho}\right)^2 s}{c^* - c_t} ds \\
p_t &= \frac{1}{r}\frac{c_t\left(c^* - c_t\right)}{c^* - c_t} - \frac{1}{c^* - c_t}\left(\frac{r\sigma}{r+\rho}\right)^2 \int_0^{\infty} s e^{-rs} ds
\end{aligned}
$$

$$
\int_0^{\infty} s e^{-rs} ds = s\frac{e^{-rs}}{-r}|_0^{\infty} - \int_0^{\infty} \frac{e^{-rs}}{-r} ds = \frac{1}{r^2}
$$

$$
p_t = \frac{c_t}{r} - \frac{1}{c^* - c_t}\frac{\sigma^2}{(r+\rho)^2}
$$

We can find the riskfree rate from the discount factor,

$$
\Lambda_t = e^{-rt}(c^* - c_t)
$$

$$
\begin{aligned}
d\Lambda_t &= -r\Lambda_t - e^{-rt} dc_t \\
E_t \frac{d\Lambda_t}{\Lambda_t} &= -r
\end{aligned}
$$

Ch2 #3 new version

3. (a)

$$
\begin{aligned}
R^f(u) &= 1/\left(\beta\pi_{u\to u}u^{-\gamma} + \beta\pi_{u\to d}d^{-\gamma}\right) \\
R^f(d) &= 1/\left(\beta\pi_{d\to u}u^{-\gamma} + \beta\pi_{d\to d}d^{-\gamma}\right)
\end{aligned}
$$

(b) The stock is exactly like the perpetuity but with $u^{1-\gamma}$ where there was $u^{-\gamma}$ and so forth.

$$
p_t = E\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(p_{t+1} + c_{t+1})\right)
$$

$$
\frac{p_t}{c_t} = E\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{p_{t+1}}{c_{t+1}}\frac{c_{t+1}}{c_t} + \frac{c_{t+1}}{c_t}\right)\right)
$$

$$
\frac{p_t}{c_t} = E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}\left(\frac{p_{t+1}}{c_{t+1}} + 1\right)\right]
$$

$$
\begin{aligned}
\frac{p}{c}(u) &= \beta\pi_{u\to u}u^{1-\gamma}\left(1 + \frac{p}{c}(u)\right) + \beta\pi_{u\to d}d^{1-\gamma}\left(1 + \frac{p}{c}(d)\right) \\
\frac{p}{c}(d) &= \beta\pi_{d\to u}u^{1-\gamma}\left(1 + \frac{p}{c}(u)\right) + \beta\pi_{d\to d}d^{1-\gamma}\left(1 + \frac{p}{c}(d)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{p}{c}(u) &= \beta\pi_{u\to u}u^{1-\gamma} + \beta\pi_{u\to d}d^{1-\gamma} + \beta\pi_{u\to u}u^{1-\gamma}\frac{p}{c}(u) + \beta\pi_{u\to d}d^{1-\gamma}\frac{p}{c}(d) \\
\frac{p}{c}(d) &= \beta\pi_{d\to u}u^{1-\gamma} + \beta\pi_{d\to d}d^{1-\gamma} + \beta\pi_{d\to u}u^{1-\gamma}\frac{p}{c}(u) + \beta\pi_{d\to d}d^{1-\gamma}\frac{p}{c}(d)
\end{aligned}
$$

$$
\left[\begin{array}{cc} 1 - \pi_{u\to u}u^{1-\gamma} & -\pi_{u\to d}d^{1-\gamma}\frac{p}{c} \\ -\pi_{d\to u}u^{1-\gamma} & 1 - \pi_{d\to d}d^{1-\gamma} \end{array}\right]\left[\begin{array}{c} \frac{p}{c}(u) \\ \frac{p}{c}(d) \end{array}\right] = \left[\begin{array}{c} \pi_{u\to u}u^{1-\gamma} + \pi_{u\to d}d^{1-\gamma} \\ \pi_{d\to u}u^{1-\gamma} + \pi_{d\to d}d^{1-\gamma} \end{array}\right]
$$

$$
\left[\begin{array}{c} \frac{p}{c}(u) \\ \frac{p}{c}(d) \end{array}\right] = \left[\begin{array}{cc} 1 - \beta\pi_{u\to u}u^{1-\gamma} & -\beta\pi_{u\to d}d^{1-\gamma} \\ -\beta\pi_{d\to u}u^{1-\gamma} & 1 - \beta\pi_{d\to d}d^{1-\gamma} \end{array}\right]^{-1}\left[\begin{array}{c} \beta\pi_{u\to u}u^{1-\gamma} + \beta\pi_{u\to d}d^{1-\gamma} \\ \beta\pi_{d\to u}u^{1-\gamma} + \beta\pi_{d\to d}d^{1-\gamma} \end{array}\right]
$$

(c) There aren't really any formulas to give here. Just program it up, e.g.

$$R_{t+1} = \frac{p_{t+1} + c_{t+1}}{p_t} = \frac{\left(\frac{p_{t+1}}{c_{t+1}} + 1\right)\frac{c_{t+1}}{c_t}}{\frac{p_t}{c_t}}$$

so

$$R_{t+1}(u \to d) = \frac{\left(\frac{p_{t+1}}{c_{t+1}}(d) + 1\right)\frac{c_{t+1}}{c_t}(d)}{\frac{p_t}{c_t}(u)}$$

and so on. Similarly

$$R^p_{t+1}(u \to d) = \frac{p^p_{t+1}(d) + 1}{p^p_t(u)}.$$

(d) You're just taking expected values of the items in c,

$$E_t(R_{t+1}|u) = \pi_{u\to u}\frac{\left(\frac{p_{t+1}}{c_{t+1}}(u) + 1\right)u}{\frac{p_t}{c_t}(u)} + \pi_{u\to d}\frac{\left(\frac{p_{t+1}}{c_{t+1}}(d) + 1\right)d}{\frac{p_t}{c_t}(u)}.$$

$$E_t\left(R^p_{t+1}|u\right) = \pi_{u\to u}\frac{p^p_{t+1}(u) + 1}{p^p_t(u)} + \pi_{u\to d}\frac{p^p_{t+1}(d) + 1}{p^p_t(u)}$$

and so forth,

(e) Here we go. I printed out some extra results here to flesh out the details.

```
Results for      u      d   beta  gamma      x    phi
              1.06   0.98   0.99   1.00   0.00   0.00

                Dc   E dC Bond p    p/c     Rf     ER    ERp  ER-Rf ERp-Rf
good state u  6.00   2.00  34.82  99.00   2.87   3.03   2.87   0.16   0.00
bad  state d -2.00   2.00  34.82  99.00   2.87   3.03   2.87   0.16   0.00

             Stock Return  stock excess  Bond Return   bond excess
 To state:        u      d      u      d      u      d      u      d
   From u:     7.07  -1.01   4.20  -3.88   2.87   2.87   0.00   0.00
   From d:     7.07  -1.01   4.20  -3.88   2.87   2.87   0.00   0.00
```

The first thing you notice is that all the prices and other forward-looking things are the same in each state. Thus the bond price, stock p/c ratio, risk free rate and expected returns are constant through time. Well, of course. Since the probabilities of u vs. d at $t+1$ are the same, everything looks the same *going forward* at t, whether you're in u or d at time t. Returns vary of course. If you go from d to u, you get a higher dividend and, since p/c is constant, a higher price too. Thus return is good to u and bad to d. Since the bond price never changes the bond return 2.87 is constant, and equal to the risk free rate. The stock expected return is a little more than the risk free rate, but not much, only 0.16%. This model is in fact the original model that launched the "equity premium puzzle," and its inability to generate a large – 6% or more – risk premium for stocks is the central puzzle.

To get some variation in things like the p/c ratio, we try increasing x to 0.3. This results in a 0.3 AR(1) coefficient for consumption growth, $\Delta c_{t+1} = a + 0.3\Delta c_t + \varepsilon_{t+1}$.

```
Results for      u      d   beta  gamma      x    phi
              1.06   0.98   0.99   1.00   0.30   0.30
```

	Dc	E dC	Bond p	p/c	Rf	ER	ERp	ER-Rf	ERp-Rf
good state u	6.00	3.20	35.04	99.00	4.10	4.24	4.04	0.15	-0.06
bad state d	-2.00	0.80	36.22	99.00	1.68	1.82	1.62	0.14	-0.06

	Stock Return		stock excess		Bond Return		bond excess	
To state:	u	d	u	d	u	d	u	d
From u:	7.07	-1.01	2.97	-5.11	2.85	6.23	-1.24	2.13
From d:	7.07	-1.01	5.40	-2.69	-0.50	2.76	-2.18	1.09

Now you see there is some variation across the states. You see the interest rate is higher in the good state than the bad state. In the good state it is more likely that *tomorrow* will be good as well, so $E_t \Delta c_{t+1}$is higher, and so $R^f \approx 1 + \delta + \gamma E_t \Delta c_{t+1}$ is higher. Since the interest rate is higher in the good state, the bond price is lower. It's only a bit lower, since the interest rate is expected to revert to its mean pretty quickly. Alas, the stock price/dividend ratio is still the same in the two states. Why? Consider the first problem – we showed that with log utility the price/consumption ratio is always constant, even if consumption growth is forecastable. The "substitution" or "discount rate" effect – higher interest rate when expected consumption growth is higher, meaning lower p/c – exactly offsets the "cashflow" effect – higher future consumption growth means p/c should rise. The expected stock and bond returns are higher in the good state, but most of this is due to the interest rate. The expected *excess* stock returns are almost the same in the two states. The expected excess bond return is slightly *negative.* Ok, bond prices are *lower* in the good state, and bond returns are thus lower if we *go to* the good state. Thus bond returns are *negatively* correlated with consumption growth. A negative beta means a negative expected excess return.

The equity premium is still troublingly low. Let's try raising the risk aversion coefficient, for example, $\gamma = 5$, reasoning that more risk aversion should give a higher risk premium. This will also break the log utility constant p/c ratio.

Results for	u	d	beta	gamma	x	phi
	1.06	0.98	0.99	5.00	0.30	0.30

	Dc	E dC	Bond p	p/c	Rf	ER	ERp	ER-Rf	ERp-Rf
good state u	6.00	3.20	12.35	14.02	15.72	15.26	14.11	-0.45	-1.61
bad state d	-2.00	0.80	14.47	15.95	3.01	2.65	1.78	-0.36	-1.22

	Stock Return		stock excess		Bond Return		bond excess	
To state:	u	d	u	d	u	d	u	d
From u:	13.56	18.42	-2.16	2.71	8.10	25.27	-7.62	9.55
From d:	-0.13	4.15	-3.14	1.14	-7.74	6.91	-10.74	3.91

The bond price is still lower in the good state, as the interest rate is higher. Both effects are larger, $R^f \approx 1 - \delta - \gamma E_t(\Delta c_{t+1})$ means more change in interest rate for a given change in expected consumption growth. We see the p/c *lower* in the good state, the opposite of what we seem to see in the data. In the data, p/c is higher in good times like the 1990s. Going back to problem 1, though, this is what we expect. For $\gamma > 1$ the discount rate effect overwhelms the cashflow effect, and we get lower p/c when expected consumption growth is high. The bond risk premium $E(R^p) - R^f$ is negative as before, but larger now. The stock risk premium is surprisingly *negative*. What's going on here? Look at the returns. The stock return is *worse* when we go to good times than when we go to bad times. As we go to good times, you get a good dividend news, and good price news if p/c is constant. But if, as here, p/c is lower in good times, that means prices go up less than consumption, and if p/c is a lot lower in good times, as here, prices can even go down, and go down enough to offset the high dividend. That's what's going on here – good times have *low* prices because they have such high interest rates. Thus, the beta or covariance of stock returns with consumption growth is *negative,* and this means the stock risk premium is negative too.

To get a positive risk premium, then, it looks like we need $x < 0$. This means that "good times" today are actually more likely to mean "bad times" tomorrow. Let's see what happens with $x = -0.3$,

```
Results for       u       d   beta  gamma       x    phi
               1.06    0.98   0.99   5.00  -0.30  -0.30

                 Dc    E dC Bond p    p/c      Rf      ER     ERp  ER-Rf ERp-Rf
good state u   6.00    0.80  10.59  12.58    3.01    4.30    3.74   1.30   0.73
bad  state d  -2.00    3.20   9.66  11.69   15.72   17.28   16.62   1.57   0.90

             Stock Return  stock excess   Bond Return   bond excess
 To state:        u       d       u       d       u       d       u      d
   From u:    14.43   -1.15   11.42   -4.15    9.45    0.66    6.44  -2.34
   From d:    23.15    6.39    7.44   -9.33   19.99   10.36    4.27  -5.36
```

Now we get some more sensible numbers. The stock p/c ratio is higher in the good state. That's good. The equity premium is up to 1.3-1.5%, which while not the 6-8% we see in the data is a lot better than 0.15%. The equity premium is positive because the p/c is high in the good state; this means that returns *to* the good state are higher than returns *to* the bad state; positive beta means positive expected excess return. The bad news is that interest rates are *low* in the good state and *high* in the bad state. Of course – in the good state now, $E_t(\Delta c_{t+1})$ is low since $x < 0$. Interest rates depend on the *future*, and the "good" state has a bad outlook for the future with $x < 0$. It is the low interest rate in the good state that gives us the high stock price. In the same way, the long term bond price is higher in the good state and earns a positive but smaller risk premium. In the data the high price/consumption ratio is associated with a low expected return. That's the right sigh, but notice it's all due to the low interest rate. The high price/consumption ratio *in the data* is associated with a low equity premium, not a low interest rate. Thus, this model is giving us way too much variation in interest rates relative to variation in expected *excess* returns.

This problem makes a deep point – betas, or the covariation with asset returns with risk factors like consumption, should not be *taken as given* or in fancy language *exogenous* to the model. Betas are part of the model too. Though you might have said "high risk aversion means a high equity premium" in this case that is false, because the *beta* turned around and became negative.

Here we are stretching beyond the current state of the art in asset pricing. Current work – for example the FF model and the CAPM – take betas as given and ask about the resulting expected excess returns. This isn't wrong, as the betas are what they are. But it leaves open the question why are the betas what they are. Ideally we should write models with cashflows (earnings, dividends, etc.) and derive the betas as well as the mean returns. I hope that 10 years from now, we'll be able to do it in a quantitatively convincing way.

You may be disappointed that I don't give you a set of parameters that "works" – that provides a slowly moving p/c ratio that forecasts returns, is high in good times and low in bad times, and that provides a nice 6% equity premium. As far as I know there are no such parameters. To reproduce these features of the data we need a new utility function. The "habit persistence" utility described in Chapter 20 is one of several fundamental changes to this model that does the trick.

This problem though long also introduces you to the fundamental *techniques* used to solve interesting asset pricing models, including the Black-Scholes model of option prices and the interest rate models such as Cox Ingersoll and Ross. In all these cases, we find a set of *state variables* like consumption growth here, that summarize everything there is to know about the future. Then, we reason that prices can only be a function of state variables, so we find the *function* relating price to state variables. In Black-Scholes the state variable is the stock price, and we find $C(S)$. In more advanced option pricing, the interest rate or level of volatility are additional state variables.

Buzzwords: you have simulated a "two-state Markov chain," solved the "Mehra-Prescott economy" and "found prices as a function of state variables by solving a functional equation."