Problem Set 1 Answers

Part I

1. To do these you exploit the fact that ε_{t+1} is not known at time t, $E_t(\varepsilon_{t+1}) = 0$ and ε_{t+1}, ε_{t+2} are uncorrelated.

$$\begin{aligned} x_{t+1} &= \rho x_t + \varepsilon_{t+1} \\ E_t(x_{t+1}) &= E\left(x_{t+1}|x_t\right) = \rho x_t + E\left(\varepsilon_{t+1}|x_t\right) = \rho x_t \\ \sigma_t^2(x_{t+1}) &= \sigma^2\left(x_{t+1}|x_t\right) = \sigma_\varepsilon^2 \end{aligned}$$

The key here is that you know x_t so when you take the mean and variance you don't have to take the mean and variance of x_t. That would be the *unconditional* mean and variance. See below. Misunderstanding conditional vs. unconditional is the biggest conceptual hurdle for time series. Continuing,

$$\begin{aligned} x_{t+2} &= \rho x_{t+1} + \varepsilon_{t+2} \\ x_{t+2} &= \rho^2 x_t + \rho\varepsilon_{t+1} + \varepsilon_{t+2} \\ E_t(x_{t+2}) &= E(x_{t+2}|x_t) = \rho^2 x_t \\ \sigma^2(x_{t+2}) &= \sigma^2\left(x_{t+2}|x_t\right) = \rho^2\sigma_\varepsilon^2 + \sigma_\varepsilon^2 = \left(1+\rho^2\right)\sigma_\varepsilon^2 \end{aligned}$$

Continuing this way,

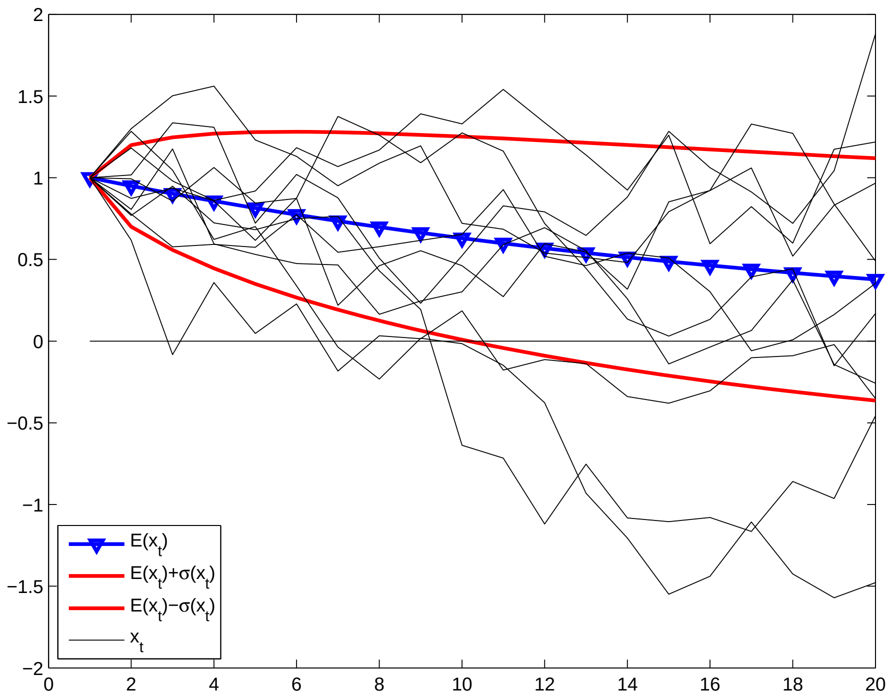
(a)

$$E_t(x_{t+j}) = \rho^j x_t$$

(b)

$$\sigma_t^2(x_{t+j}) = (1 + \rho^2 + .. + \rho^{2(j-1)})\sigma_\varepsilon^2 = \frac{1-\rho^{2j}}{1-\rho^2}\sigma_\varepsilon^2.$$

(c)



(d) The unconditional mean is

$$E(x_t) = E_{t-\infty}(x_t) = E_t(x_{t+\infty})$$

so we can just take limits, and the same for variance. In this case

$$\begin{aligned} E(x_t) &= 0 \\ \sigma^2(x_t) &= \frac{\sigma^2}{1-\rho^2} \end{aligned}$$

What about $\rho = 1$? Yes, then the variance goes to infinity. That is a "random walk" and it really does have infinite unconditional variance.

(e)

$$(x_t - \mu) = \rho(x_{t-1} - \mu) + \varepsilon_t$$

or

$$x_t = (1-\rho)\mu + \rho x_{t-1} + \varepsilon_t.$$

2.

(a) $E\left[r_{t+1} + r_{t+2} + .. + r_{t+k}\right] = kE(r) = k\mu$; $E\left(\frac{1}{k}\left[r_{t+1} + r_{t+2} + .. + r_{t+k}\right]\right) = E(r) = \mu$. The mean long-horizon return should grow with the horizon. The mean annualized return should be the same at any horizon. The mean monthly return is 0.5% the annualized mean monthly return is 6%.

(b) $\sigma^2(r_{t+1} + r_{t+2} + .. + r_{t+k}) = k\sigma^2(r)$. The variance of returns should grow with horizon. $\sigma^2\left[\frac{1}{k}(r_{t+1} + r_{t+2} + .. + r_{t+k})\right] = \frac{1}{k}\sigma^2(r)$. The variance of *annualized* returns goes down with horizon. The assumption that r are independent over time means you can ignore the covariance terms $cov(r_t r_{t+j})$.

(c) $\sigma\left[r_{t+1}+r_{t+2}+..+r_{t+k}\right] = \sqrt{k}\sigma(r) = \sqrt{k}\sigma(r)$. The standard deviation of returns should grow with the square root of horizon. $\sigma\left(\frac{1}{k}\left[r_{t+1}+r_{t+2}+..+r_{t+k}\right]\right) = \frac{1}{\sqrt{k}}\sigma(r)$. Notice that the actual return is getting more volatile but the annualized return is getting more stable. This is the "fallacy of time-diversification." $\sqrt{0.16^2/12} = .0462$ or 4.62% and $\sqrt{0.16^2/365} = 0.00848 = 0.8\%$.

(d) 12% and $\sqrt{12}\times 1 = 3.46\%$.

(e) $E(r_{t+1}+..+r_{t+k})/\sigma(r_{t+1}+..+r_{t+k}) = \sqrt{k}\mu/\sigma$ *Sharpe ratios should scale with the square root of horizon. Yes.*

(f) $E(r_{t+1}+..+r_{t+k})/\sigma^2(r_{t+1}+..+r_{t+k}) = \mu/\sigma^2$ *The ratio of mean to variance is the same at all horizons. With iid returns, stocks are no more or less attractive to "long horizon" investors.*

(g) If we look at annualized returns, it looks like Sharpe ratios (or mean/variance) are growing with horizon, making stocks more attractive for the long run.

$$
\begin{aligned}
E\left[\frac{1}{k}(r_{t+1}+..+r_{t+k})\right]/\sigma\left[\frac{1}{k}(r_{t+1}+..+r_{t+k})\right] &= \sqrt{k}\frac{\mu}{\sigma};\\
E\left[\frac{1}{k}(r_{t+1}+..+r_{t+k})\right]/\sigma^2\left[\frac{1}{k}(r_{t+1}+..+r_{t+k})\right] &= k\frac{\mu}{\sigma^2}.
\end{aligned}
$$

Looking at annualized returns makes it look like the long run is safer, when it's not (See "fallacy of time-diversification" in the notes.)

3.

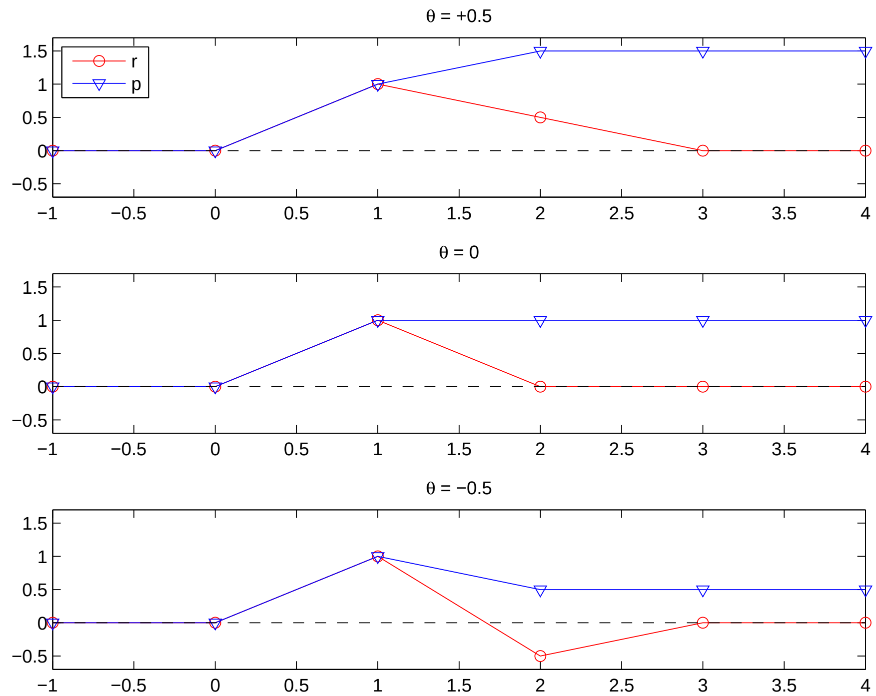
(a)

$$
\begin{aligned}
r_t &= \mu+\varepsilon_t+\theta\varepsilon_{t-1}\\
E(r_t) &= \mu\\
var(r_t) &= \sigma_\varepsilon^2+\theta^2\sigma_\varepsilon^2 = \left(1+\theta^2\right)\sigma_\varepsilon^2\\
cov(r_t,r_{t-1}) &= cov(\varepsilon_t+\theta\varepsilon_{t-1},\varepsilon_{t-1}+\theta\varepsilon_{t-2})\\
&= \theta\sigma_\varepsilon^2\\
\theta\sigma_\varepsilon^2 &= \frac{\theta}{1+\theta^2}var(r_t)\\
cov(r_t,r_{t-j}) &= 0
\end{aligned}
$$

Note again in taking the variance how you cleverly use the fact that $cov(\varepsilon_t,\varepsilon_{t-1}) = 0$. The impulse-response for returns is

$$1,\theta,0,0,0.$$

The impulse response for "price" is just the cumulative sum. Notice that when returns are positively correlated, $\theta > 0$, then prices keep going up after a shock – "momentum." When returns are negatively correlated, $\theta < 0$, then prices "bounce back" after a shock, "mean-reversion."



(b)

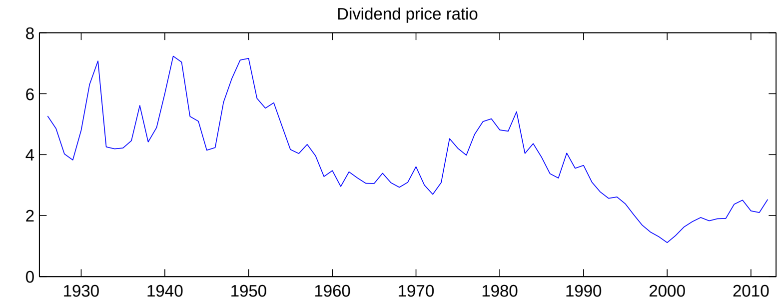
$$E\left[r_{t+1}+r_{t+2}+..+r_{t+k}\right]=kE(r);$$

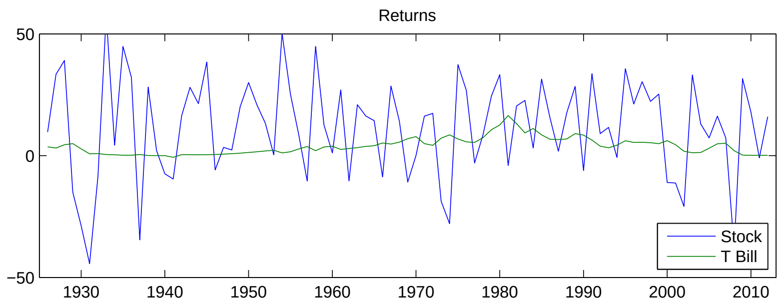
$$\begin{aligned}\sigma^2\left[r_{t+1}+r_{t+2}+..+r_{t+k}\right] &= kvar(r)+2(k-1)cov(r_t,r_{t-1})\\ &= \left[k+2(k-1)\frac{\theta}{1+\theta^2}\right]\sigma^2(r)\end{aligned}$$

The mean is unaffected. $\theta > 0$ (momentum) makes long variance larger. $\theta < 0$ (mean reversion) makes it smaller. I chose the MA(1) here because there is only one covariance term to deal with.

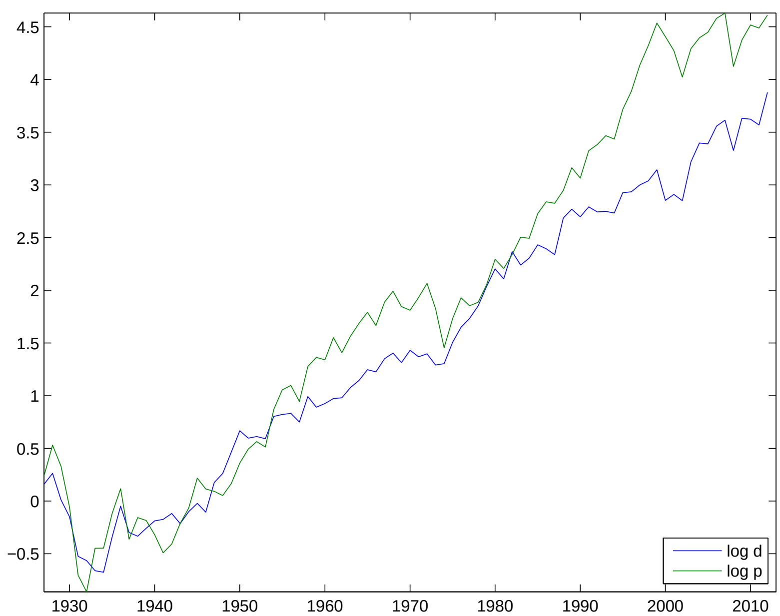
Regression problems. My program ps1.m is on the class website. If you're puzzled how I did anything, that's the place to look. Even if you don't program in matlab, the syntax is pretty transparent and you'll be able to see how I did things.

1.





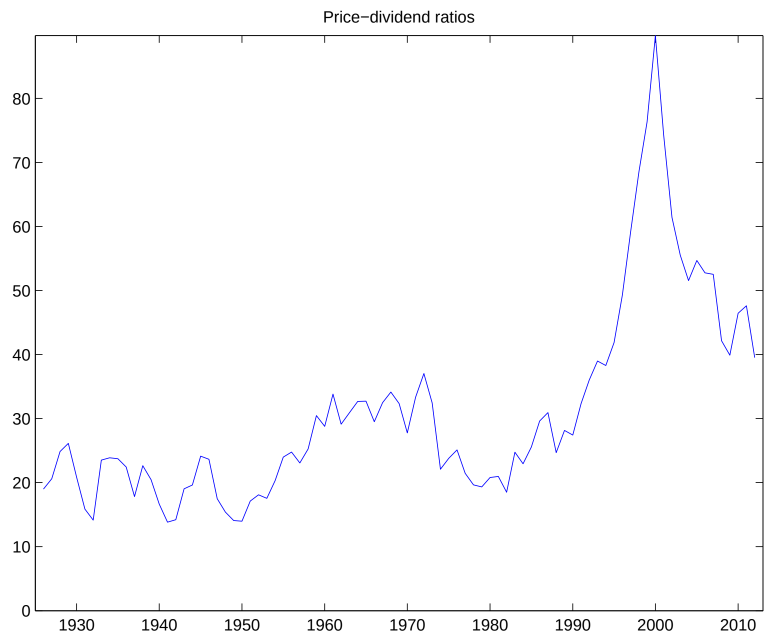
You can see the slow swings in d/p, mostly mirroring generation-long swings in prices. The stock return is much more volatile than the treasury bill return! Here is a plot (not requested) of the level of prices and dividends. As you can see, there is a lot of common movement. Disasters like 1929, 1974 and 2008 are disasters for both prices and dividends. But there is a tendency for prices to advance over dividends in good times and conversely, giving the variation you see above in the d/p ratio.



You may ask "where is the financial crisis?" A decline of log prices from 4.5 to 4.0 is a 50% decline. It's there. Here is a plot of the p/d ratios, which dramatizes things a bit. As you can see, despite the recent market declines of the early 2000s, and 2008, the pd as of 2008 when the data ends was still above historical p/d norms. What about the financial crisis? Dividends fell like a stone too, so D/P didn't move as much as you might have thought.

Here's what I said at the top in 2007:

> Three things are bound to happen 1) We will return to the normal 25 p/d ratio. This can happen either if a) prices fall by about half or b) dividends grow like they've never grown before to catch up with prices. The point of this problem set and lecture is that in the past, it's always been the prices that adjust. Good luck to us. 2) We could sit at a permanently higher d/p ratio. This could be a once in 400 year shift upwards in p/d. That still means expected returns somewhat lower than in the past 400 years, but not as bad as reversion to a 25 pd would mean.



2. Here are my results:

Regression of returns on lagged returns 1926-2012

	b	t(b)	R2	E(R)	s(E(R))
Stock	0.01	0.06	0.00	11.62	0.13
T Bill	0.92	20.42	0.83	3.95	3.16
Excess	0.01	0.11	0.00	7.66	0.25

As you can see the stock return is essentially completely uncorrelated over time. The bill return has an autocorrelation coefficient of 0.92, a huge t and R^2. That's there to emphasize that finance doesn't say *all* returns should be unpredictable. I wanted you to see that it *could* come out differently; that stock return unpredictability is a fact, not a definition. Usually we think of stock returns as an interest rate, which *can* be predictable over time (as the T bill return so clearly is) plus a risk premium that should *not* be predictable over time in the classic random-walk view of the world,

$$R_t = R_t^f + R_t^e.$$

If excess returns are predictable, you can borrow, invest in stocks and make a costless profit. If interest rates are predictable, all you can do is save more/less, which is a weak corrective force. That's why we usually focus on the risk premium part, $R^e_t = R_t - R^f_t$, to focus on the part that does not contain even the small predictable component coming from bonds. As you see in the third row, the excess return is also unpredictable. $\sigma(E_t(R_{t+1})) = 0.25\%$ means that if you believe $b \neq 0$, the market risk premium only varies by about 25 bp ($\sigma(E(R))$)over time, very small compared to the 11.62% historical premium.

3-4. Here are my results

Regression of excess returns on lagged D/P 1926-2012

	b	t(b)	R2	E(R)	s(E(R))	E(R)/k	s(E(R))/k
1 Yr.	3.68	2.54	0.07	7.66	5.54	7.66	5.54
5 Yr. overlap	19.03	4.99	0.24	44.94	28.47	8.99	5.69
5 Yr. no overlap	21.61	2.52	0.30	46.62	34.42	9.32	6.88

You can see a dramatic difference. In the one year return, we now have a 2.5 t statistic and a 0.07 R^2. The coefficient is big too. A coefficient of 1 means that prices don't move: one percent more dividend yield means one percent more return. A coefficient of 3.7 means that prices move the "wrong way," reinforcing the dividend yield.

The 5 year coefficient 5 times larger than the one year coefficient. As in class, this results because d/p forecasts R_{t+1}, R_{t+2}. etc. about the same way.

Amazingly $\sigma(E_t(R))$ is as big as $E(R)$ itself! If you believe this, the equity premium varies over time by about its average magnitude, jumping between 0 and 12, not sitting at 7.5%.

The overlapping regression b and R^2 are not biased. The standard error is biased; we effectively have 1/5 as many data points. (If the return from 1-5 got lucky, the return from 2-6 mechanically shares some of that luck.)

Formally, in a regression $y_t = a + bx_t + \varepsilon_t$, the OLS assumptions are $cov(x_t, \varepsilon_t) = 0$, and $cov(\varepsilon_t \varepsilon_{t-j}) = 0$. Our regression is $R^{\text{Excess}}_{t,t+5} = a + b\,(D/P)_t + \varepsilon_{t+5}$. It's still true that $cov(D/P_t, \varepsilon_{t+5}) = 0$. It's still a forecast error so should be unpredictable. It is not true that $\text{cov}(\varepsilon_{t+5}.\varepsilon_{t+6}) = 0$ however, because they have 4 observations in common. When we do regressions with no overlap, notice we get roughly the same coefficient and R^2 but much less t stat. It's still significant though.

The non-overlapping data regression is ok on both counts. Notice how it gives just about the same coefficient and R^2 (also unaffected by overlap) but much different t statistic.

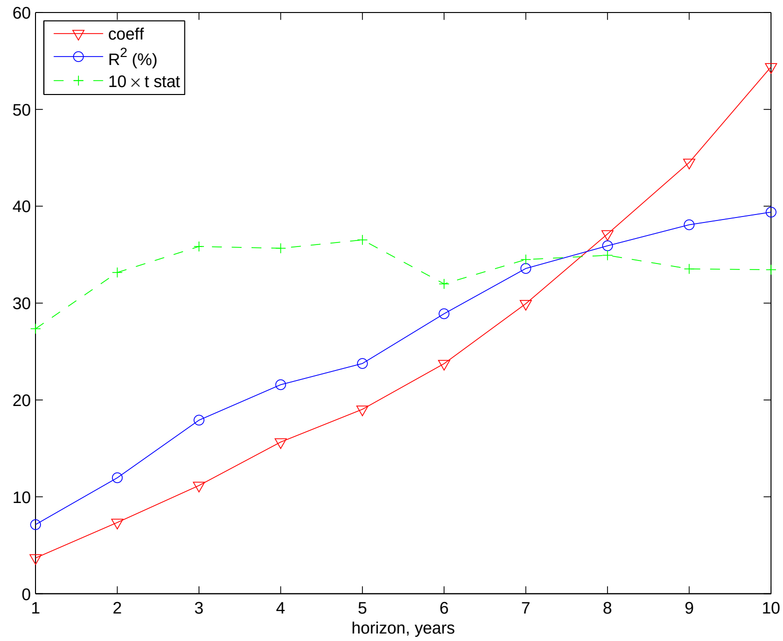
Since the overlapping data *coefficients* are unbiased, you can use overlapping data for coefficients but use the nonoverlapping regression standard errors. Better, there is a formula for how to produce correct standard errors. (The correction formula is a bit complicated, it's the "Hansen-Hodrick" correction described in *Asset Pricing* if you want to see it.)

FYI here is a table and a graph of regressions of 1-10 year horizon. Here I used overlapping data but I corrected the standard errors for the overlap using the "Hansen-Hodrick correction". My program olsgmm on the class website does this correction.

NYSE excess returns on D/P, overlapping, many horizons, 1926-2012

Horizon (years)

	1	2	3	4	5	6	7	8	9	10
b	3.68	7.35	11.17	15.66	19.03	23.74	29.93	37.13	44.52	54.39
t	2.74	3.32	3.58	3.57	3.65	3.20	3.45	3.49	3.35	3.34
R2	0.07	0.12	0.18	0.22	0.24	0.29	0.34	0.36	0.38	0.39



Notice the b coefficients rising about linearly with horizon. The t's are about the same, and the R^2 increase with horizon. These are all patterns we talked about in class. It's not as good as the coefficients and regressions in *Asset pricing* because the late-90s price explosion despite already high prices. Quote from last year's problem set: "A few more years of poor returns and we'll be back on track!"

d. Here's my plot. I include the 10 year horizon too. As you see the d/p tracks well these long-term movements in expected returns. The 1990s were a major disappointment to the forecast, but we seem back on track now.