Business 35904 **John H. Cochrane**

Problem Set 2

Due Wed Oct 30

1. Here we will examine the firm's problem, in discrete and continuous time. The firm wakes up with capital k_t. It earns $\theta_t k_t$, and pays for investment i_t out of retained earnings, and distributes profits to investors. There is an adjustment cost, so that larger investment costs output. While the painters are in your office, you don't get a lot of work done. Thus, if the firm buys investment goods i_t it pays for them, so profit goes down by $-i_t$, and in addition it loses output in the amount $c(i_t/k_t)i_t$. Effectively, the price of the investment good is higher when the firm is doing a lot of investment. (Pretty much everything we will do here can quickly be generalized to a nonlinear form, replacing $\frac{\alpha}{2}\left(\frac{i}{k}\right)$ with $c\left(\frac{i_t}{k_t}\right)$.) The firm maximizes its contingent claim value. (If you've seen maximizing value discounted at a risk free rate, that is the nonstochastic special case. This is the right objective.)

$$\begin{aligned}
\pi_t &= \theta_t k_t - \left[1 + \frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)\right] i_t \\
V_t(k_t, \cdot) &= \max_{\{i_t\}} E_t \sum_{j=0}^{\infty} m_{t+j}\pi_{t+j} \\
s.t.\ k_{t+1} &= (1-\delta)(k_t + i_t)
\end{aligned}$$

I write $\cdot$ for information variables at time t, for example variables that describe the distribution of discount factors and productivity shocks.

The value $V(k_t, \cdot_t)$ is also the market value of the firm at the *beginning* of period t, i.e. including period t profits. Our usual timing has stock price at the end of the period. Thus, define $W_t = V_t - \pi_t = E_t\left(\frac{\Lambda_{t+1}}{\Lambda_t}V_{t+1}\right)$, and then the financial return to owning the stock is the conventional $R_{t+1} = (W_{t+1} + \pi_{t+1})/W_t = V_{t+1}/(V_t - \pi_t)$. "Q" or Market/book ratio in finance is the ratio of market value to book value, $Q_t = W(k_t, \cdot_t)/(k_t + i_t)$. (Q is measured at the end of the period, so the capital stock ready to go for the next period, before depreciation, is $k_t + i_t$.) This is a one-good economy, so Q represents the relative value of "installed" vs. "uninstalled" capital.

Your task: derive the "Q theory of investment" that links investment to stock prices, and derive the "investment return" linking the stock rate of return to investment in each period. (This is basically a quasi-differenced version of the Q theory.) There are lots of ways to do this. This is only a suggestion, and I'm not sure it's the cleanest approach. If you can find a cleaner way to get the same results, do it. In particular, if you're familiar with stochastic optimization, this is a pretty trivial application of the Hamiltonian. However, I think you miss a lot of intuition going that way.

   (a) Imagine raising i_t and leaving the rest of the plan alone. Find the resulting first order condition linking investment i_t to the stream of future productivity, investment, capital, etc. Hint: $\partial k_{t+j}/\partial i_t = (1-\delta)^j$.

   (b) From this answer, find the scaling law. If $\{i_t, k_t\}$ is optimal, given k_0, what is the optimal sequence $\{i_t, k_t\}$ given $2k_0$? If $V(k_0, \cdot_t)$ is the value function given k_0, what is the value function $V(2k_0, \cdot_t)$?

   (c) Now, write the problem in recursive form, $V(k_t, \cdot_t) = \max_{\{..\}}(..._t) + ...V(k_{t+1}, \cdot_{t+1})\ s.t. k_{t+1} = (1-\delta)(k_t + i_t)$. Find the first order condition linking investment today to time $t+1$ value function derivatives. Then, find the relationship between investment and the time t stock price and Q_t. (You can also reason by envelope theorem, that an extra dollar has the same value whether invested or taken as profit)

(d) Thinking ahead, macroeconomists will want to test this "theory of investment," and financial economists will want to test this "investment-based theory of stock prices." What regression would you run or what moment condition would you examine to test this model? What is the source of the error term in this relationship? Is it a forecast error? What is the error orthogonal to?

(e) Having expressed investment as, essentially, a function of stock price, let us now describe returns. We're looking for return on the left and a function of investment and capital at $t+1$ and t on the right. Again, there are many ways to do this. Here are a few suggestions.

   i. From the relationship between investment and stock price, create the financial rate of return and express it as a function of investment and capital at time t and $t+1$. Try to make it pretty, as dimensionless as possible, i.e. using i/k where possible.

   ii. Suppose the firm raises i_t and then lowers i_{t+1} just enough to leave profit the same at $t+2$ and thereafter. The one period return is the amount that profit π_{t+1} is raised divided by the amount that profit π_t declines. This operation creates a physical return – money goes in at t and goes out at $t+1$, and no money goes in or out at later dates. Figure out that return. And, step 2, show that return is equal to the stock market return, ex post, data point for data point, as defined above. If you get the same answer as last time, you know you got it right. Doing it this way you also see how the physical investment return is related to the stock return.

(f) The consumption-based model generated a link between consumption and asset returns

$$1 = E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R^i_{t+1}\right]$$

which we could estimate and test by GMM, or by running regressions

$$\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R^i_{t+1} = 1 + bz_t + \varepsilon_{t+1}$$

and looking for $b=0$. What regression would we run for the investment return model, which links investment returns to investment/capital ratios? What probability measure do we use for firms? Is the relation between investment and returns a pure parallel with the consumption relation, or is it essentially different somehow?

(g) What happens in this model if $\alpha = 0$?

2. The timing conventions must have driven you nuts. They drove me nuts, which is why this problem set was not ready on Monday. Restate the same thing in continuous time. The corresponding continuous-time problem is

$$\begin{aligned}
V_t(k_t,\cdot) &= \max_{\{i_t\}} E_t \int_{s=0}^{\infty} \frac{\Lambda_{t+s}}{\Lambda_t}\pi_{t+s}ds \\
s.t. dk_t &= (-\delta k_t + i_t)\, dt \\
\pi_t &= \theta_t k_t - \left[1 + \frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)\right] i_t
\end{aligned}$$

You don't have to be formal about this, but

(a) Write down the relationship between investment and stock price. You can reason by analogy with discrete time, or realize that an extra dollar of capital or an extra unit of profit should have the same vale.

(b) As we did in discrete time, from the definition of return given stock price, write the relationship between investment, capital and the instantaneous return on the firms' stock. Hint: like consumption, i will follow an ito process, k_t will not. Timing will be easy, but ito terms will surface.