Problem set 2 answers

1. (a)

$$1+\alpha\frac{i_t}{k_t}=E_t\sum_{j=1}^{\infty}m_{t+j}(1-\delta)^j\left\{\theta_{t+j}+\frac{\alpha}{2}\left(\frac{i_{t+j}}{k_{t+j}}\right)^2\right\}$$

(b) This is a constant returns technology. $V(k_t,\cdot)=k_tV(1,\cdot_t)$. This means $\partial V(k_t,\cdot)/\partial k_t = V(k_t,\cdot_t)/k_t$.

(c)

$$\begin{aligned}V(k_t,\cdot) &= \max_{\{i_t\}}\left\{\theta_t k_t-\left[1+\frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)\right]i_t\right\}+E_t\left[m_{t+1}V(\cdot_{t+1})\right]\ s.t.\ k_{t+1}=(1-\delta)\left(k_t+i_t\right)\\ 1+\alpha\frac{i_t}{k_t} &= (1-\delta)E_t\left[m_{t+1}\frac{\partial V(k_{t+1},\cdot)}{\partial k_{t+1}}\right]=(1-\delta)E_t\left[m_{t+1}\frac{V_{t+1}}{k_{t+1}}\right]\\ &= (1-\delta)\frac{W_t}{k_{t+1}}=(1-\delta)\frac{W_t}{(1-\delta)\left(k_t+i_t\right)}=\frac{W_t}{\left(k_t+i_t\right)}=Q_t.\end{aligned}$$

$$1+\alpha\frac{i_t}{k_t}=Q_t.$$

$$\frac{i_t}{k_t}=\frac{1}{\alpha}\left[Q_t-1\right].$$

(d) There is no error! This model produces an exact relationship between investment and Q. You can run a regression with either variable on the left hand side because there should be no error! Of course in reality there is an error, which we need to call "measurement error" or really "specification error." There is no reason to believe such errors are orthogonal to anything. This is a good case to meditate on "quantitative parables" vs. "theories you test by maximum likelihood."

i.

$$\begin{aligned}R_{t+1} &= \frac{W_{t+1}+\pi_{t+1}}{W_t}=\frac{Q_{t+1}(i_{t+1}+k_{t+1})+\theta_{t+1}k_{t+1}-\left[1+\frac{a}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]i_{t+1}}{Q_t(i_t+k_t)}\\ &= \frac{\left[1+\alpha\frac{i_{t+1}}{k_{t+1}}\right](i_{t+1}+k_{t+1})+\theta_{t+1}k_{t+1}-\left[1+\frac{a}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]i_{t+1}}{\left[1+\alpha\frac{i_t}{k_t}\right](i_t+k_t)}\\ &= (1-\delta)\frac{\left(\left[1+\alpha\frac{i_{t+1}}{k_{t+1}}\right](\frac{i_{t+1}}{k_{t+1}}+1)+\theta_{t+1}-\left[1+\frac{a}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]\frac{i_{t+1}}{k_{t+1}}\right)k_{t+1}}{\left[1+\alpha\frac{i_t}{k_t}\right]k_{t+1}}\\ R_{t+1} &= (1-\delta)\frac{1+\theta_{t+1}+\alpha\left(\frac{i_{t+1}}{k_{t+1}}\right)+\frac{\alpha}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2}{1+\alpha\left(\frac{i_t}{k_t}\right)}\end{aligned}$$

Intuition: you pay the adjustment cost to get in. Then capital depreciates. You get the marginal product, and the effect of greater capital in reducing next period's adjustment costs. Then you get to reduce investment next period, and get the benefit of not paying adjustment costs.

ii. The firm increases investment by di_t at time t. That lowers profits by $\partial\pi_t/\partial i_t=1+\alpha\left(\frac{i_t}{k_t}\right)$. Next period, the firm has $dk_{t+1}=(1-\delta)di_t$ more capital stock, which generates profits $\partial\pi_{t+1}/\partial k_{t+1}=\theta_{t+1}+\frac{\alpha}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2$. Now it also can lower i_{t+1} so as to leave k_{t+2} unchanged,

$di_{t+1} = -(1-\delta)di_t$ which generates additional profit $\partial\pi_{t+1}/\partial i_{t+1} = \left[1+\alpha\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]$ In sum, the investment return is

$$R^I_{t+1} = (1-\delta)\frac{\theta_{t+1} + \frac{\alpha}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2 + 1 + \alpha\left(\frac{i_{t+1}}{k_{t+1}}\right)}{1+\alpha\left(\frac{i_{t+1}}{k_{t+1}}\right)}.$$

The algebra of the last answer proves that this is the asset return. You invest, and pay adjustment costs. Then your capital depreciates. You get the marginal product of capital, the adjustment costs themselves are lower, in the second term. The most important term is the "price change" term linear in (i_t/k_t). To first order,

$$R^I_{t+1} = 1-\delta+\theta_{t+1}+\alpha\left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{t+1}}{k_{t+1}}\right)$$

and since $k_{t+1} \approx k_t$,

$$R^I_{t+1} \approx 1-\delta+\theta_{t+1}+\alpha\left(\frac{i_t}{k_t}\right)\left(\frac{i_{t+1}-i_t}{i_t}\right)$$

(e) Again, there is no error. This is a pure arbitrage relationship between the physical investment opportunities and the financial investment opportunities. That is a result of this particular, common, but totally arbitrary production function specification.

(f) Endogenously, if $\alpha = 0$ then $Q = 1$. *Without adjustment costs the book/market ratio is always one. There is no possibility of price variation! All returns come from changing marginal products of capital/profits only. To match data with large valuation effects, we need to have adjustment costs.*

2. This on

$$\begin{aligned} -\frac{\partial\pi_t}{\partial i_t} &= \frac{\partial V_t}{\partial k_t} \\ 1+\alpha\left(\frac{i_t}{k_t}\right) &= \frac{\partial V_t}{\partial k_t} = \frac{V_t}{k_t} = Q_t \end{aligned}$$

$$dR_t = \frac{dV_t + \pi_t dt}{V_t}$$

$$1+\alpha\left(\frac{i_t}{k_t}\right) = \frac{V_t}{k_t}$$

$$\begin{aligned} V_t &= k_t + \alpha i_t \\ dV_t &= dk_t + \alpha di_t = (i_t - \delta k_t)dt + \alpha di_t \end{aligned}$$

You can see right away that di_t will match the price volatility, uncertainty part,

$$\frac{dV_t}{V_t} = \frac{(i_t-\delta k_t)dt + \alpha di_t}{k_t+\alpha i_t} = \frac{(\frac{i_t}{k_t}-\delta)dt + \alpha\left(\frac{i_t}{k_t}\right)\frac{di_t}{i_t}}{1+\alpha\frac{i_t}{k_t}}$$

$$\frac{\pi_t}{V_t}dt = \frac{\theta_t k_t - \left[1+\frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)\right]i_t}{k_t+\alpha i_t}dt = \frac{\theta_t - \left[1+\frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)\right]\frac{i_t}{k_t}}{1+\alpha\left(\frac{i_t}{k_t}\right)}dt$$

$$dR_t = \frac{(\frac{i_t}{k_t} - \delta)dt + \alpha\left(\frac{i_t}{k_t}\right)\frac{di_t}{i_t} + \theta_t dt - \left[1 + \frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)\right]\frac{i_t}{k_t}dt}{1 + \alpha\frac{i_t}{k_t}}$$

$$dR_t = \frac{\left[\theta_t - \delta - \frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)^2\right]dt + \alpha\left(\frac{i_t}{k_t}\right)\frac{di_t}{i_t}}{1 + \alpha\left(\frac{i_t}{k_t}\right)}$$

Compare to the discrete time version

$$R_{t+1} = (1-\delta)\frac{1 + \theta_{t+1} + \frac{\alpha}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2 + \alpha\left(\frac{i_{t+1}}{k_{t+1}}\right)}{1 + \alpha\left(\frac{i_t}{k_t}\right)}$$

In the continuous time version the economics is oh so much clearer: you pay adjustment costs to get in, then there is a cashflow term, then there is a price change term where investment growth is proportional to stock price growth.