Problem Set 5 Answers

Part I A simple very short readings questions

1.
$$R^{ep}_{t+1} = R^{ei}_{t+1} - \beta_{i1} f^1_{t+1} - \beta_{i2} f^2_{t+1} = \alpha_i + \varepsilon^i_{t+1}.$$
$$\frac{E\left(R^{ep}_{t+1}\right)}{\sigma\left(R^{ep}_{t+1}\right)} = \frac{\alpha_i}{\sigma(\varepsilon^i)}$$

2. Yes, like temperature. See the plot of utility in the notes. Marginal utility should be positive.

3. $p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] = \beta E_t\left[x_{t+1}\right]$ If people are risk neutral, prices are just a constant times expected payoffs.

4.
$$\begin{aligned} m_t &\approx e^{-\delta} e^{-\gamma \Delta c_{t+1}} = e^{-0} e^{-0} - e^{-0} e^{-0} \delta - \gamma e^{-0} e^{-0} \Delta c_{t+1} + \frac{1}{2} \gamma^2 e^{-0} e^{-0} \Delta^2 c_{t+1} \\ &\approx 1 - \delta - \gamma \Delta c_{t+1} + \frac{1}{2} \gamma^2 \Delta c^2_{t+1} \end{aligned}$$

5.
$$\frac{\|E(R^e)\|}{\sigma(R^e)} \leq \frac{\sigma(m)}{E(m)} \approx \gamma \sigma(\Delta c) = 0.02 \times 10 = 0.20.$$
That's a problem, since the market Sharpe ratio is about 8/16=0.5. This is called the equity premium puzzle.

Part I B

1. The plot is below.

   (a)
$$\begin{aligned} \ln(c_t) + \beta \ln(c_{t+1}) &= k \\ \ln(c_{t+1}) &= \frac{k - \ln(c_t)}{\beta} \\ c_{t+1} &= e^{\frac{k - \ln(c_t)}{\beta}} \end{aligned}$$

   I set up a grid for c_t, a couple of different k chosen artistically, and set matlab to work on it.

   (b) The slope is in general
$$\begin{aligned} d\left[U(c_t, c_{t+1}) - k\right] &= 0 \\ \frac{\partial U}{\partial c_t} dc_t + \frac{\partial U}{\partial c_{t+1}} dc_{t+1} &= 0 \\ \frac{dc_{t+1}}{dc_t} &= -\frac{\partial U / \partial c_t}{\partial U / \partial c_{t+1}} \\ \frac{dc_{t+1}}{dc_t} &= -\frac{1/c_t}{\beta / c_{t+1}} = -\frac{c_{t+1}}{\beta c_t} \end{aligned}$$

   The third to last equation is important – indifference curve slope = ratio of marginal utilities = marginal rate of substitution. At $c_{t+1} = c_t$ the slope is 1/0.95 ≈ -1.05. So the slope is a bit steeper than -1.

(c) My solution is a little more general; I wait until the end to plug $u(c) = \ln(c)$ and the numbers in to the answer. You probably got there quicker by specializing earlier. You also might have avoided the Lagrangian by substituting, e.g. $c_{t+1} = RW - c_t$.

$$\begin{aligned} & \max_{\{c_t, c_{t+1}\}} u(c_t) + \beta u(c_{t+1}) \\ s.t. W &= c_t + \frac{1}{R} c_{t+1} \end{aligned}$$

$$L = u(c_t) + \beta u(c_{t+1}) - \lambda \left[c_t + \frac{1}{R} c_{t+1} - W \right]$$

$$\begin{aligned} \frac{\partial L}{\partial c_t} &: \quad u'(c_t) - \lambda = 0 \\ \frac{\partial L}{\partial c_{t+1}} &: \quad \beta u'(c_{t+1}) - \frac{1}{R}\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &: \quad c_t + \frac{1}{R} c_{t+1} - W = 0 \end{aligned}$$

From the first two,

$$u'(c_t) = \beta R u'(c_{t+1})$$

In the log case

$$\begin{aligned} \frac{1}{c_t} &= \beta R \frac{1}{c_{t+1}} \\ c_{t+1} &= \beta R c_t \end{aligned}$$

Substituting into the budget constraint,

$$\begin{aligned} c_t + \frac{1}{R} (\beta R c_t) &= W \\ c_t &= \frac{W}{1+\beta} = \frac{1000}{1.95} = \$512.82 \\ c_{t+1} &= \beta R \frac{W}{1+\beta} = 0.95 \times 1.10 \times \frac{1000}{1.95} = \$535.90 \end{aligned}$$

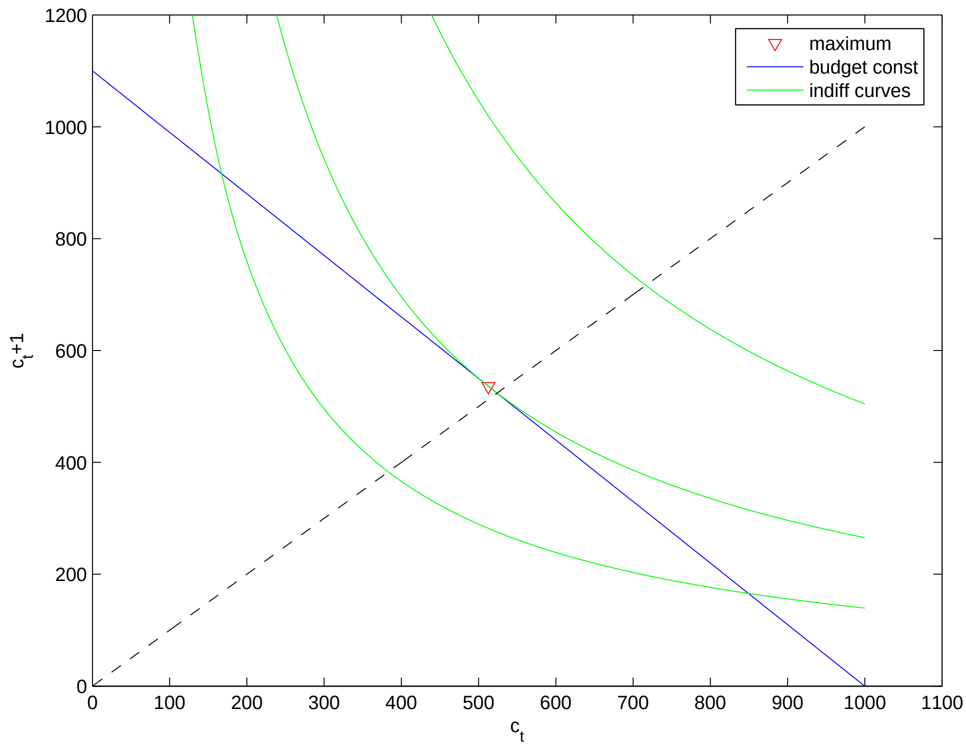
In order to plot the associated indifference curve, I found the k for this optimal choice

$$\ln(c_t^*) + \beta \ln(c_{t+1}^*) = \ln(512.82) + 0.95 \times \ln(535.90) = 12.21$$

The budget constraint is of course

$$\begin{aligned} c_t + \frac{1}{R} c_{t+1} &= W \\ c_{t+1} &= R(W - c_t) = 1.10\,(1000 - c_t) \end{aligned}$$

Now we have all the ingredients for The Plot:



(d) Notice that $c_{t+1} = \beta R c_t$, the investor consumes $\beta R = 0.95 \times 1.10 \approx 5\%$ more in the second period. The attraction of the 10% interest rate outweighs the impatience of $\beta = 0.95$, and the investor is induced to save.

2. The point here is that the relative risk aversion coefficient is a measure of how willing people are to take a gamble.

(a)

i.

$$Eu(c) = \frac{1}{2}u(\bar{c} + \bar{c}\varepsilon) + \frac{1}{2}u(\bar{c} - \bar{c}\varepsilon)$$

$$\begin{aligned} Eu(c) \quad &\approx \quad \frac{1}{2}\left[u(\bar{c}) + u'(\bar{c})\bar{c}\varepsilon + \frac{1}{2}u''(\bar{c})\bar{c}^2\varepsilon^2\right] + \\ &\quad\;\; \frac{1}{2}\left[u(\bar{c}) - u'(\bar{c})\bar{c}\varepsilon + \frac{1}{2}u''(\bar{c})\bar{c}^2\varepsilon^2\right] \end{aligned}$$

$$Eu(c) - u(\bar{c}) = \frac{1}{2}u''(\bar{c})\bar{c}^2\varepsilon^2$$

ii.

$$\begin{aligned} Eu(c) \quad &= \quad E\left[u(\bar{c} + \delta\bar{c})\right] \approx u(\bar{c}) + \delta\bar{c}u'(\bar{c}) \\ Eu(c) - u(\bar{c}) \quad &= \quad \delta\bar{c}u'(\bar{c}) \end{aligned}$$

iii.

$$-\frac{1}{2}u''(\bar{c})\bar{c}^2\varepsilon^2 = \delta\bar{c}u'(\bar{c})$$

$$-\frac{1}{2}\frac{u''(\bar{c})\bar{c}}{u'(\bar{c})} = \frac{\delta}{\varepsilon^2} = \frac{\text{bribe}}{\text{variance}}$$

Note $\sigma^2(c) = \frac{1}{2}\bar{c}^2\varepsilon^2 + \frac{1}{2}\bar{c}^2\varepsilon^2$;so $\sigma^2(c/\bar{c}) = \sigma^2((c-\bar{c})/\bar{c}) = \varepsilon^2$. Meanwhile, $E[(c-\bar{c})/\bar{c}] = \delta$. So we can restate the answer, "how much extra mean do you need to compensate you for taking on variance," with both mean and variance expressed as percentages of expected consumption

$$-\frac{1}{2}\frac{u''(\bar{c})\bar{c}}{u'(\bar{c})} = \frac{E((c-\bar{c})/\bar{c})}{\sigma^2((c-\bar{c})/\bar{c})}$$

(b)

$$\begin{aligned} u(c) &= \frac{c^{1-\gamma}}{1-\gamma} \\ u'(c) &= c^{-\gamma} \\ u''(c) &= -\gamma c^{-\gamma-1} \\ -\frac{cu''(c)}{u'(c)} &= \gamma \end{aligned}$$

That's why we call γ the risk aversion coefficient! The bribe is $1/2\gamma$, but we usually leave out the 1/2.

Part II A

1. Here are my regressions.

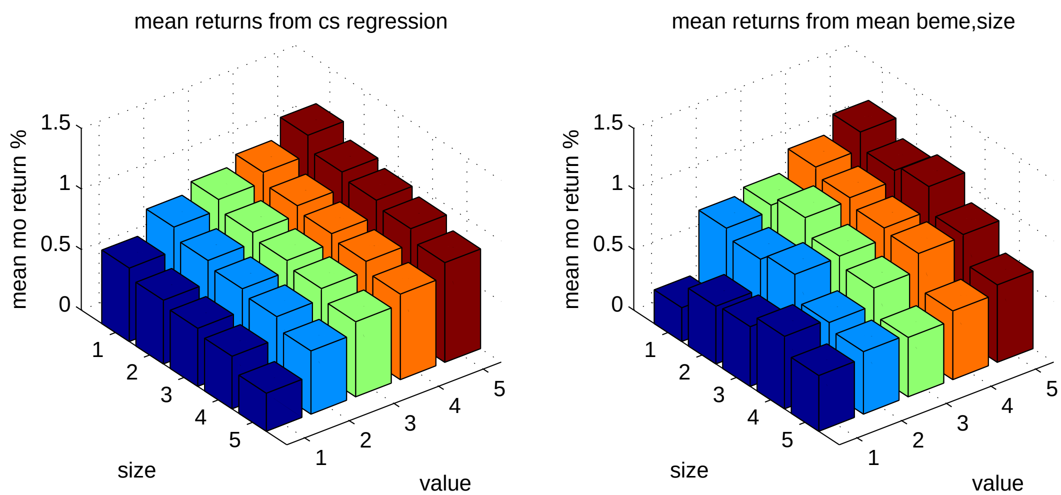
```
Data sample
     196301.00      201312.00
regression of mean xs return on mean log beme and mean log size
          const      size      beme        R2
    b     0.616    -0.052     0.266     0.761
```

As expected, more size means less returns, more beme means higher returns. These are about the same size coefficients as reported by FF Table 4. A 10% increase in size means a -5bp decrease in monthly return; a 10 percent increase in beme (i.e. beme from 1.0 to 1.10) means a 27 bp increase in monthly return.

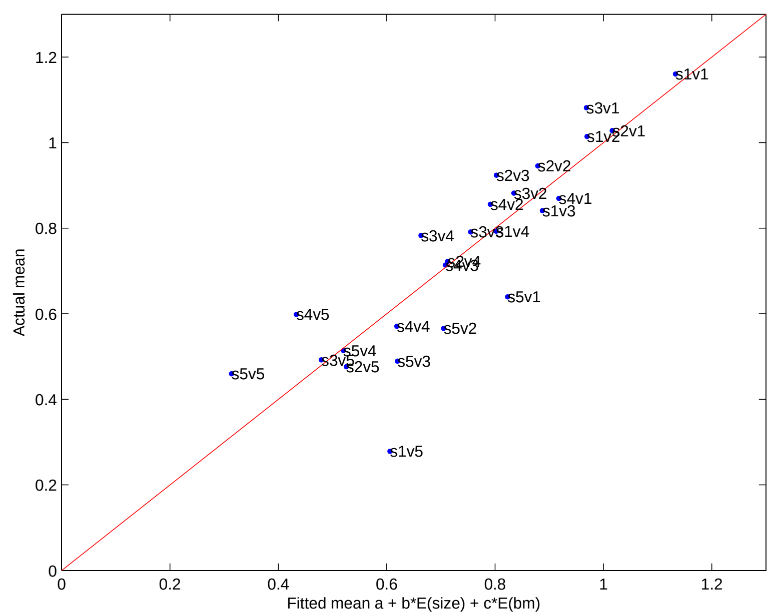
2. Here's the table.

```
actual mean return
          0.28          0.79          0.84          1.01          1.16
          0.48          0.72          0.92          0.95          1.03
          0.49          0.78          0.79          0.88          1.08
          0.60          0.57          0.71          0.86          0.87
          0.46          0.51          0.49          0.57          0.64
regression on average log size and average log beme
          0.61          0.80          0.89          0.97          1.13
          0.53          0.71          0.80          0.88          1.02
          0.48          0.66          0.75          0.83          0.97
          0.43          0.62          0.71          0.79          0.92
          0.31          0.52          0.62          0.71          0.82
```

As usual, it's clearer in a plot:



I also made another plot of fitted vs. actual means.



As you can tell in the numbers and graph, the cross sectional regression is doing a pretty good job. This plot looks very much like the mean return plot we started with. If the regression were perfect, we'd see a 45 degree line. The major failing is, no surprise, s1v5, the small growth portfolio. This portfolio does much worse than a linear extrapolation of "being small" and "being growth" suggests it should.

3. The regression with a cross term.

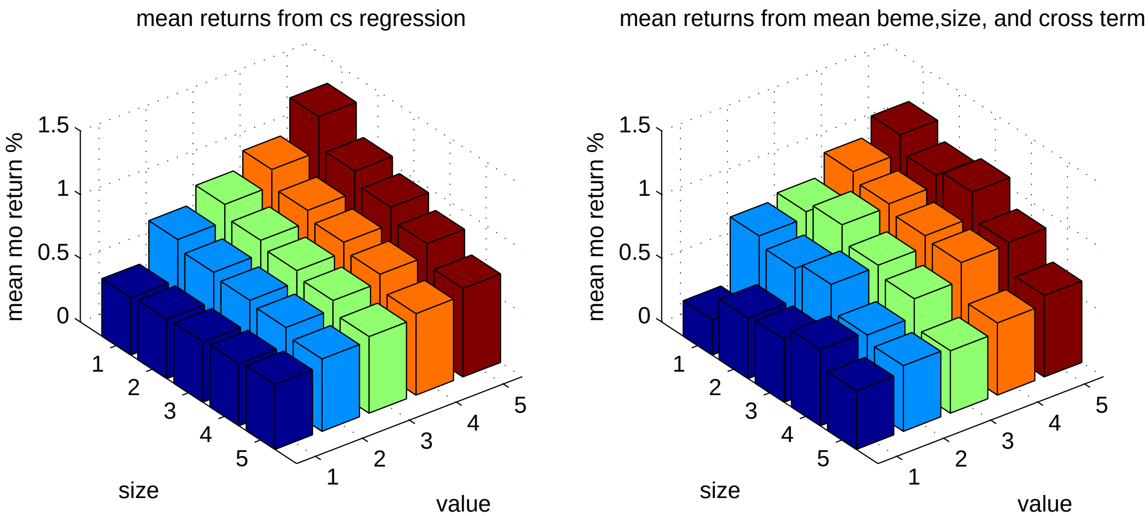
$$E(R^{ei}) = a + b \times E(size_i) + c \times E\left(bm_{it}\right) + d \times E(size_i) \times E\left(bm_{it}\right) + \varepsilon_i$$

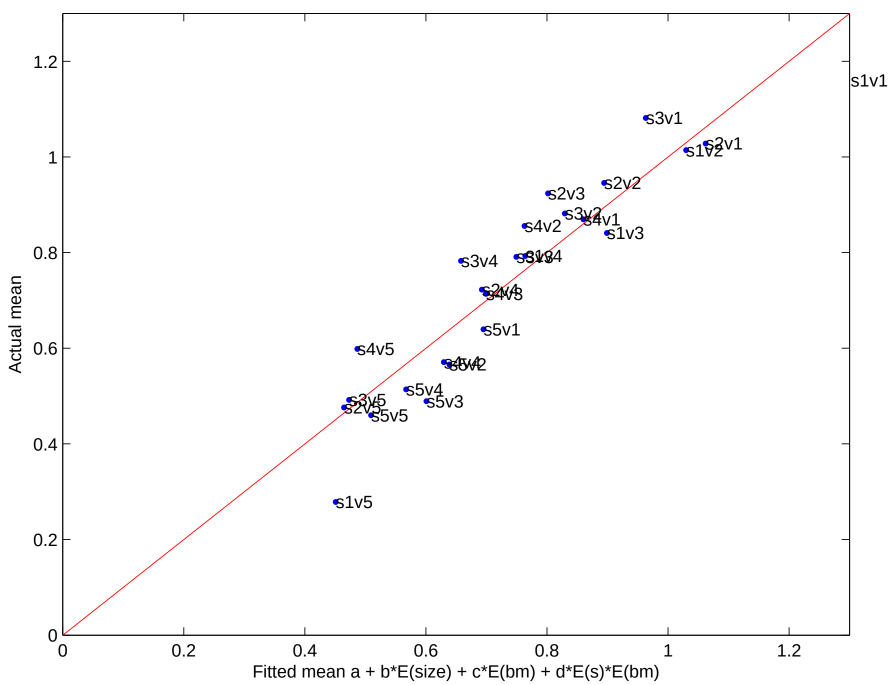
regression of mean xs return on mean log beme and mean log size and cross term

	const	size	beme	s x bm	R2
b	0.480	-0.081	-0.039	-0.068	0.870

One measure of success, the R^2 has gone up to 0.87 from 0.75. You can see how with the cross term the dark blue fitted value can go down for growth stocks and up for value stocks, and give therefore a much better fit. One way to see what this does, is that it allows a greater size effect for value than for growth stocks. I.e.

$$E(R^{ei}) = a + [b + d \times E(bm_{it})] \times E(size_i) + c \times E(bm_{it}) + \varepsilon_i$$





Though not perfect, you can see that the pattern of Table 1A is not really just a "size" effect and a "beme" effect, but there is an important "interaction" as well. The betas will have to have the same pattern if they are to explain these returns.

4. Yes, you are allowed to do anything you want in these regressions Remember, *we're just "describing returns," we're not "explaining" returns here.*. The content of these regressions is simply whether the pattern in Table 1 panel A can be matched by a *linear* function of suitably transformed B/M and size. If so, that is a convenience, but no theory of anything hangs on the answer. Table 1A itself is a 25 parameter nonlinear fit. Is a linear function of transformed BM and size a good *description* of average returns? Pretty much yes, except for the small growth anomaly. But unlike betas, there is no economic reason why average returns *should* be linear functions of BM and size. We can use nonlinear functions, or cross terms if we want.

By contrast you are not allowed to use cross terms or nonlinear functions in

$$E(R^{ei}) = \alpha_i + b_i\lambda_m + s_i\lambda_s + h_i\lambda_h + (s_i \times h_i)\lambda_{sh} + \varepsilon_i; \quad i = 1, 2, ...25$$

The factor model says average returns depend linearly on betas. Period. That's because there is a deep portfolio interpretation. We form the portfolio

$$R_t^{ei} - b_i rmrf_t - s_i smb_t - h_i hml_t$$

You can't do that with cross terms.

5. A reminder: the cross sectional regression

```
regression of mean xs return on mean log beme and mean log size
          const     size     beme       R2
    b     0.616   -0.052    0.266     0.761
```

Now, the forecasting regressions

```
pooled forecasting regression of return on log beme and log size
          const     size     beme       R2
    b     0.655   -0.060    0.451    0.003
```

You see roughly the same coefficient on size. The beme coefficient is the same sign but about twice as big. Why?

```
pooled forecasting regression of sample mean x for each portfolio
     b    0.616   -0.052    0.266
pooled forecasting regression of x - sample mean x for each portfolio
     b    0.740   -1.028    1.465
```

Aha! *The forecast regression using the sample mean gives exactly the same result as the cross sectional regression!* But we see in both coefficients that if portfolio A has an unusually high b/m, relative to portfolio A's long-term average, that is a signal that A will have a strong return the next year, and this signal is even stronger than the signal that portfolio A has, on average, a higher b/m than portfolio B. This is what FF left on the table. We can do better by looking at change in value over time, not just across stocks.

This is a "pooled regression with portfolio dummes." Taking out the portfolio averages – and thus looking only at variation over time in the right hand variables

$$R_{t+1}^{ei} = a + b \times [size_{it} - E(size_{it})] + c \times [bm_{it} - E(bm_{it})] + \varepsilon_{it+1}.$$

is *mechanically* the same as putting in portfolio dummies

$$R^{ei}_{t+1} = a_i + b \times size_{it} + c \times bm_{it} + \varepsilon_{it+1}.$$

Additional discussion (not in the problem set)

You can also do the opposite. Let's take out the average size and BM *across portfolios at each moment in time.* This is the same thing as adding time dummies,

$$R^{ei}_{t+1} = a_t + b \times size_{it} + c \times bm_{it} + \varepsilon_{it+1}.$$

Now the regression *ignores* variation over time, but only asks if portfolio i has more than portfolio j, how does that affect returns

```
pooled forecasting regression, time means
    b   -14.038   -3.510    2.153
pooled forecasting regression, removing time means
    b     0.740   -0.051    0.289
```

In this problem you have run a "cross-sectional regression" and a "pooled time-series cross-sectional regression."

Part B

1.

```
          const     size     beme
Cross sectional regression, regress_jc t statistics
    b     0.616   -0.052    0.266
   se     0.068    0.014    0.037
    t     9.021   -3.801    7.136
Conventional s,t statistics
conv se   0.065    0.013    0.036
    t     9.422   -3.970    7.453
Corrected se, t statistics using Sigma/T
corr se   0.193    0.037    0.081
    t     3.185   -1.424    3.286
Fama MacBeth coefficients, standard errors, t statistics
 FMB b    0.587   -0.050    0.244
    se    0.203    0.037    0.080
    t     2.889   -1.346    3.055
```

We saw the regression above. The T statsitics are huge, boy these effects are strong.

The "conventional" statistics $(x'x)^{-1}\sigma^2(e)$ use the variance of the residual, where the regression package uses 1/(N-k) in place of 1/N. As you see this makes a small difference in t statistics. I hope you remembered to take a square root

The corrected t statistics are half or less of the uncorrected statistics! In cross-sectional regressions with thousands of individual stocks, a factor of 10 is more common. *Do not run cross sectional regressions* *with t statistics that are not corrected for cross-sectional correlation.* It is one of the most common mistakes even in published articles.

The FMB coefficients are just about the same as the cross-sectional coefficients! There is close to a theorem here. If the x variables are the same over time (if we had used $E(x)$ not x_t) then FMB and cross sectional coefficients would have been *exactly* the same. Variation of x_t over time gives rise to the small difference.

The FMB standard errors and ts are just about the same as the corrected ones! You can see a great similarity in the way its constructed. FMB was invented before this formula was invented, but survives because everyone knows it so well.