Business 35150 **John H. Cochrane**

Problem Set 6

Due Saturday Week 6 12:00

Part I Reading Questions

Give short answers.

Carhart.

The introduction summarizes his conclusions:

- Momentum in *stocks* accounts for momentum in *funds*. Funds that did well last year have stocks that went up and those stocks will keep rising a bit. It is *not* persistent skill, or good returns for momentum funds. Momentum funds do poorly after transactions costs. There is some persistent *under* performance. Important: Survivor bias free data – includes funds that die. (Lots of hard work by Carhart, and another great CRSP dataset.) (p. 58)

We need to look for the facts! *Find the facts behind these assertions in the paper.*

Now *Questions*

1. On p.61 Carhart defends the four-factor model as a *performance attribution* model.
   (a) Why is it OK to use a "momentum factor" even if that is not a "state variable for investment opportunities?"
   (b) What *question* are we using the multifactor model to answer, and how is that question different from Fama and French's question?
2. (Hint: Table III is the most important. Spend most of your time to understand it.) How does Carhart form portfolios of mutual funds - -what are Portfolio 1A 1B...10C in column 1 of Table III?
3. Do the funds that went up last year always continue to go up? How much risk is there in this investment strategy? To quantify these questions, what is the chance that portfolio 1A will earn a positive return in a typical month of next year?
4. How do the CAPM R^2 values compare to those for stocks you have seen before? What accounts for the difference?
5. Are all the alphas zero after the 4 factor model is done, or is there a puzzle? Who seems still to be outperforming and who is underperforming?
6. Fund managers claim that fees and turnover do not reduce returns to investors. How could charging more money *not* reduce returns to investors? (Try to be a good salesperson for a high-turnover high-fee fund. Why should I give you my money? Then try to be a good supply-demand economist. What should the equilibrium relationship be between fees, expenses and returns to investors?)
7. (Table V. Make sure you understand how this table was created. How are Table IV and Table V different?) What does Carhart find about fees and turnover? How much does a 1% change in fees change returns to investors? How much does turnover – selling one stock and buying another – change returns to investors?

8. What is the point of Figure 2? (Hint: what would it look like if the sort on one year performance indicated skill?)

9. What does Carhart say about momentum funds – funds that seem to follow a momentum strategy, as revealed by high loadings on the momentum factor? How do we know that we're not seeing the performance of momentum funds? (Hint: no table, but text on p. 73.)

10. *One* year lagged returns are probably mostly luck, not skill. What if you sort funds by the more common 5 year performance averages? (Hint: Figure 3)

Fama and French Mutual Fund Performance

So far, we have been looking for "skill" by guessing some characteristic associated with skill – past returns, MBA by manager, etc. – and looking at the return of a sorted portfolio going forward. This paper tells us whether there is any skill at all, *without* us taking a stand on what characteristic can be used to find good funds. It answers the question "sure the average fund is mediocre, but there are some good funds." Read 1916 top to understand why they're different than persistence tests – if there is skill, lagged returns are a very noisy measure of that skill.

1. What do Fama and French mean by "Equilibrium Accounting? (p. 1915 top)

2. Fama and French focus on the alpha t statistic. Why not look at alphas or information ratios?

3. Explain the numbers in Table 3.

   (a) What does the 95 row, first two columns (95 1.68 1.54) mean? (Hint: At what number x is the probability that a $N(0,1)$ is larger than x is 5%?)

   (b) Why is the probability of a t greater than 2 or less than -2 not the usual 5% value that we expect for a t statistic?

4. Why can't we explain fat tails of estimated alphas by fat tails of the return distribution?

5. Do funds look better using only the CAPM in Table AI? IF so, what to FF say about it?

Berk

1. What happens to future returns and flows, according to Berk, if a manager does have some skill?

2. Berk says, unlike FF, that managers do have some skill even though alphas are all zero. How can that be?

3. Berk says that when investors chase past returns, investing in funds that have done well in the past, they are not being irrational, even though future returns are no better than average. How can this be?

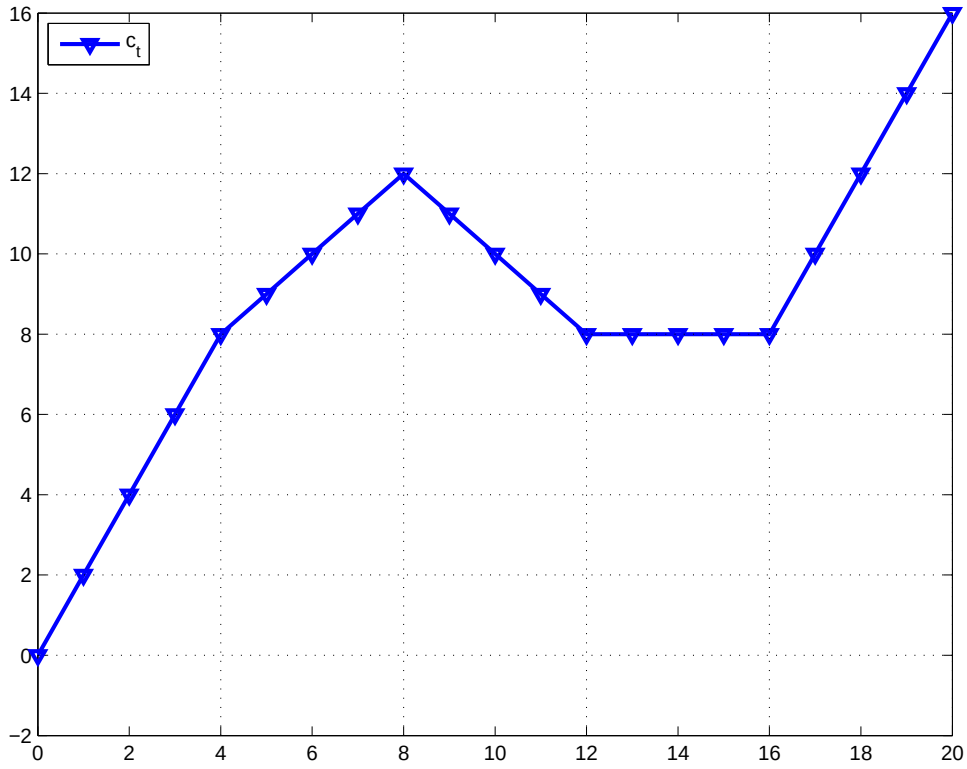
4. Berk says that even though skill is permanent, returns will not be persistent. Why not?

Part II Consumption Problems

1. (From a previous final.) The graph represents consumption over time, in percent (100 x log).

   (a) Use the consumption-based model to find and plot the interest rate r_t^f over time, also in percent, assuming people know ahead of time where consumption Δc_{t+1} is going. Use discount rate $\delta = 2\%$, and risk aversion $\gamma = 2$, and approximate as necessary to get round (integer) answers. Hint: Make sure you put the interest rate at the right moment in time. t vs. $t+1$ is vital here!

   (b) How do interest rates correlate with booms and busts, measured by levels of consumption and by differences in consumption?



   Note: If you want to do this with a computer rather than a pen, you can create my graph with the following code and then add consumption growth and interest rates to the graph

```
tim = (0:20)';
c = [ 0 2 4 6 8 9 10 11 12 11 10 9 8 8 8 8 8 10 12 14 16 ]';
figure
plot(tim, c,'-v','linewidth',2);
hold on
plot([1;1]*(4:4:20),[-2 16],':k');
plot([0;20],[1;1]*(0:2:14),':k');
legend('c_t',2);
```

2. Suppose an investor has a "habitual" level of consumption X, and really does not want consumption to fall below X. ("I'd rather die than fly commercial, honey" -overheard at hedge fund conference.)

Now the utility function is

$$u(c) = \frac{(c-X)^{1-\gamma}}{1-\gamma}$$

(The same situation occurs if X represents borrowing, like mortgage payments, and there are huge costs of bankruptcy if the investor does not repay the loan. This utility function can also represent a university endowment that has to pay X to its tenured professors, a defined-benefit pension fund that must pay X to retirees, or a leveraged hedge fund that must pay its debt. Throughout this problem assume $c > X$.)

(a) Plot this utility function, and compare it to the standard $X = 0$ case $u(c) = c^{1-\gamma}/(1-\gamma)$. (My plot uses $\gamma = 2$, $X = 1$. A freehand sketch is fine too.)

(b) Recall we defined risk aversion as $\eta \equiv -\frac{cu''(c)}{u'(c)}$ and we derived that $\eta = \gamma$ for the power utility function. (This is a good time to check that fact.) What is the risk aversion coefficient $\eta = -\frac{cu''(c)}{u'(c)}$ for this investor? Plot risk aversion as a function of consumption using $\gamma = 2$. Include the $X = 0$ case as well in your plot.

(c) To capture the latter intuition in a formula, recall that we derived

$$E_t(R^{ei}_{t+1}) \approx \gamma cov(R^{ei}_{t+1}, \Delta c_{t+1})$$

for power utility. This formula generalizes when risk aversion is not equal to γ, $\eta \neq \gamma$ as

$$E_t(R^{ei}_{t+1}) \approx \eta_t cov(R^{ei}_{t+1}, \Delta c_{t+1})$$

(Derivation[1]). I use the subscript t, $\eta_t = -c_t u''(c_t)/u'(c_t)$, to emphasize that if risk aversion varies, it's risk aversion at time t that governs how the investor feels about holding returns from time t to time $t+1$.

Now, imagine a security whose return moves one for one with consumption growth; if you need an equation $R^e_{t+1} = E_t(R^e_{t+1}) + (\Delta c_{t+1} - E_t(\Delta c_{t+1}))$. (This is the problem set version of the "market portfolio.") In the power utility case, we would find the expected return on this asset by

$$E_t(R^{ei}_{t+1}) \approx \gamma cov(R^{ei}_{t+1}, \Delta c_{t+1}) = \gamma \sigma^2(\Delta c_{t+1})$$

Find the expected return of this security with the habit X, and contrast it with the power utility case. In the power case, the expected return is constant over time. In the habit case, does the expected return $E_t(R^e_{t+1})$ rise, fall or stay the same if consumption c_t has fallen?

(Assume that the mean and standard deviation of consumption growth $E_t(\Delta c_{t+1}) = E(\Delta c_{t+1})$ and $\sigma_t(\Delta c_{t+1}) = \sigma(\Delta c_{t+1})$ are constant over time. The tricky part here is getting c_t and c_{t+1} straight. In this problem, losses from time $t-1$ to t affecting c_t affect risk aversion at time t, which in turn affects how the investor evaluates the risk of holding returns from time t to time $t+1$. I assigned the problem on purpose to have you sort out confusion about time t vs. time $t+1$ consumption.)

(d) I want to think about what happens to asset *prices* in a recession. The right way to answer this question is to work out $p_t = E_t \sum_{j=1}^{\infty} \beta^j u'(c_{t+j})/u'(c_t) D_{t+j}$, and academic articles on this utility function do that. Too much math for a problem set. Instead, let's do a simpler

[1]

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \approx 1 - \delta + \frac{c_t u''(c_t)}{u'(c_t)} \frac{(c_{t+1} - c_t)}{c_t}$$

The right hand side is a Taylor expansion around $\delta = 0$ and $c_{t+1} = c_t$. This is all exact, and much prettier, in continuous time. So,

$$m_{t+1} = 1 - \delta - \left(-\frac{c_t u''(c_t)}{u'(c_t)} \right) \frac{(c_{t+1} - c_t)}{c_t}$$

version. Let's imagine a security that pays a dividend c_{t+1}, so its expected return will be the one we just calculated, and let's approximate its asset price as

$$P_t = \frac{E_t(c_{t+1})}{E_t\left(R^e_{t+1}\right)}$$

and thus the "price dividend ratio" as

$$\frac{P_t}{c_t} = \frac{E_t(c_{t+1}/c_t)}{E_t\left(R^e_{t+1}\right)}.$$

In the power utility case, we would then write

$$\frac{P_t}{c_t} = \frac{1}{\gamma}\frac{E_t(\Delta c_{t+1})}{\sigma^2(\Delta c_{t+1})}.$$

We note that the price-consumption ratio is constant, so if c_t falls 10%, then prices also fall 10% – the "cashflow shock" we found in our VAR.

Continuing the example from the last question, then, write the price-consumption ratio in the habit case.

(e) Let's work out an example. Suppose c_t falls 10%, $c_t/c_{t-1} = 0.90$. Suppose and X was 80% of initial consumption c_{t-1}, $(c_{t-1} - X)/c_{t-1} = 0.2$, so now X is 90% of consumption c_t, i.e. $(c_t - X)/c_t = 0.10$. (Yes, I'm approximating.) How much does the price/consumption ratio rise or fall? How much do prices rise or fall? (In percent.) Does the habit induce "stabilization" or "amplification" or shocks?

(f) This was all about time t and expected return. Let's think about time $t+1$ and covariances. When we studied the consumption-based model with power utility, we argued that

$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = e^{-\delta-\gamma\Delta c_{t+1}} \approx 1 - \delta - \gamma\Delta c_{t+1}$$

and from the linear factor model trick, we concluded that expected excess returns should be proportional to consumption betas. Repeat with this new model, and argue that we get a *two* factor model with growth in the log "surplus consumption ratio" $[(c_{t+1} - X)/c_{t+1}] \,/\, [(c_t - X)/c_t]$ as the second factor.

Hint: for power utility we wrote

$$m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = e^{-\delta}e^{-\gamma\log(\frac{c_{t+1}}{c_t})} \approx 1 - \delta - \gamma\log\left(\frac{c_{t+1}}{c_t}\right)$$

so write $m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$ in this case, express it as

$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}(x_{t+1})^{-\gamma} = e^{-\delta}e^{-\gamma\log(\frac{c_{t+1}}{c_t})-\gamma\log(x)} \approx 1-\delta-\gamma\log\left(\frac{c_{t+1}}{c_t}\right)-\gamma\log(x_{t+1}).$$

From this expression alone you can conclude that the two factors consumption growth and x drive average returns. Your job: Find x.

(g) Show how to use the return and change in price-consumption ratio as the two factors in place of consumption growth and growth in surplus consumption ratio. I.e. show how we should see a two-factor model with the return on the market portfolio and the change in valuation ratio ("book/market") of the market portfolio as separate factors.

For power utility, we could wite

$$m_{t+1} = 1 - \delta - \gamma\log\left(\frac{c_{t+1}}{c_t}\right) = a - \gamma R^e_{t+1}$$

where R^e_{t+1} is the security that moves one for one with consumption growth. (a, not $1-\delta$, because the mean might not be the same as the mean of consumption growth. We're not worrying about log vs. level of consumption growth here.) So your job, figure out how to get rid of x_{t+1} in the last problem with information on the price-consumption ratio of our asset in the same way.

This is an important problem. I think it captures a lot of what happened in the fall of 2008. Many investors have leverage or backstop commitments (mortgages) or even an accustomed level of consumption X. As they lose money, they become more risk averse and try to sell. But we can't all sell, so markets go down further. There's nothing "irrational" about it – if you're leveraged, you have to scale back after a loss. This is the heart of my story for time varying risk premiums ("By force of habit" on the optional readings.) It also means that a substantial amount of risk premium generated by betas on discount rate changes, not cashflow changes.

3. The "binomial model" is really useful in asset pricing. Suppose that there are two states tomorrow, "up" and "down," and each can happen with probability 1/2. Consumption is $c_t = 1$ today, and $c_{t+1} = 2$ in the "up" state and $c_{t+1} = 1/2$ in the "down" state. Assume $\gamma = 1$, $\beta = 1$, and calculate the following

   (a) Find the price of a bond – an asset that pays 1 in each state

   (b) Find the price of an asset that pays $x = 1$ in the up state and $x = -1$ in the down state.

   (c) Find the price of an asset that pays $x = -1$ in the up state and $x = 1$ in the down state.

   (d) Compare b and c. Which of the assets has greater mean payoff $E(x)$? Greater variance of payoff $\sigma(x)$? Explain why they differ in price.

   (e) Find the price of an asset that pays off one unit in the up state, and zero units in the down state, and the price of an asset that pays of zero units in the up state and one unit in the down state. These are "contingent claims." Which is more valuable? Why?

   (f) Now, rather than value the asset in part b directly, let's value it by arbitrage. Find the number of contingent claims from part e that replicate the asset of part b. Find the price of the replicating portfolio. Do you get the same answer?

4. We don't always use consumption growth to find a discount factor. In option pricing, we find a discount factor that prices the stock and bond ("what must consumption growth have been to make the stock and bond price what they are?") and use that to price an option. In this problem you get to see this approach.

   A stock right now ($t = 0$) has price S_0. At time $t = 1$ it will either rise to $S_1 = S_u = u\, S_0$ or decline to $S_1 = S_d = d\, S_0$ with equal probability. (u and d are numbers, like 1.06 and 0.98. Assume $u > d$ and $u > 1, d < 1$) There is also a bond that pays R^f.

   (a) Find a discount factor which prices the stock and bond by construction. What this means is, find a value for m in the "up" state m_u and m in the "down" state m_d so that $S_0 = E(mS_1) = \frac{1}{2}m_u S_u + \frac{1}{2}m_d S_d$ (that's what "E" means) and $1 = E(mR^f)$. Note: m is a random variable, not a number. When we "choose m" that means, "choose the two numbers m_u and m_d"

   (b) Use this discount factor to price an at the money call option. The option pays $C_1 = \max(S_1 - X, 0)$ with $X = S_0$ (at the money). Find its value by $C_0 = E(mC_1)$.

   (c) Find the call option value the traditional way. Set up a portfolio of k shares of stock and h dollars invested in the bond. Choose k and h to match the option payoff, $kS_1 + hR^f = \max(S_1 - X, 0)$. The value of the option is then $kS_0 + h$. You should get the same result as you did in part b.

(d) Now, we'll follow the "risk-neutral pricing" approach. Ignore the fact that the actual probabilities of the two states are each $\pi_u = \pi_d = 1/2$, and instead make up "risk neutral probabilities π_u^* and $\pi_d^* = 1 - \pi_u^*$ such that

$$S_0 = \frac{1}{R^f} E^*(S_1) = \frac{1}{R^f} \left[\pi_u^*(uS_0) + \pi_d^*(dS_0)\right]$$

Use the risk neutral probabilities to value the option,

$$C_0 = \frac{1}{R^f} E^*(C_1)$$

Do you get the same answer?

(e) How are the risk-neutral probabilities π^* related to actual probabilities π and marginal utility m? Hint: "Risk aversion is the same thing as a distorted probability assessment, in which unpleasant states are treated as if they are more likely." Can you map that statement to your equations?

(f) (Optional: to be valid probabilities we also need $0 \leq \pi_u^* \leq 1$ and $0 \leq \pi_d^* \leq 1$. Can these be violated by your formulas? If so why and how do we fix it?)

The point of this problem is to see in detail how "relative pricing" works. You do not ask why the stock and bond price are what they are in terms of things like consumption or macroeconomic risk. You just construct a discount factor that prices them, and then use that discount factor to price the option. The APT works the same way. This is the basis of the derivation of the Black-Scholes formula given in the book.

5. Risk sharing. There are two consumers with power utility but different utility functions, one with $\gamma_A = 1$ and the other with $\gamma_B = 2$. Thus, $m_{t+1}^A = \beta \left(\frac{c_{t+1}^A}{c_t^A}\right)^{-\gamma_A}$ $m_{t+1}^B = \beta \left(\frac{c_{t+1}^B}{c_t^B}\right)^{-\gamma_B}$. As in the last question, assume that markets are "complete" meaning there is a single m_{t+1} derivable from asset markets, and hence $m_{t+1}^A = m_{t+1}^B = m_{t+1}$. However, unlike the previous problem, there are lots and lots of states, so c_{t+1}/c_t (A or B) can take on lots an lots of values.

   (a) What is the relationship of log consumption growth $\log\left(c_{t+1}^A/c_t^A\right)$ to $\log(c_{t+1}^B/c_t^B)$? Who has greater mean vs. volatility of log consumption growth?

   (b) Would this outcome be different if the investors had income $y_t^A, y_{t+1}^A, y_t^B, y_{t+1}^B$, and the income growth was not correlated, y_{t+1}^A/y_t^A is uncorrelated with y_{t+1}^B/y_t^B? Is there a way for them to share this risk?

6. The *claim to the consumption stream* is a fun security to think about, and one that we often use to stand in for the stock market in theoretical models, rather than write down production, capital, labor, profits, etc., imagine a security that pays a dividend c_t at date t, c_{t+1} at date $t+1$, and so forth. In this problem we'll also get a little practice with continuous time (deterministic, not stochastic – no Ito's lemma.) Use power utility $u(c) = c^{1-\gamma}/(1-\gamma)$ and hence $u'(c) = c^{-\gamma}$; and use $\beta = e^{-\delta}$. Assume consumption growth is steady, a constant $G = \frac{c_{t+1}}{c_t} = e^g$ for all times, so $c_t = e^{gt}c_0$. In this model the E_t are superfluous as people know everything ahead of time. But I'll write them in anyway.

   (a) Find the log (continuously compounded) one-year interest rate in terms of g, δ, and γ. Is the interest rate larger or smaller when consumption growth g is larger or smaller?

   Hint: start with $1 = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} R^f\right]$ of course, and $r^f = \log(R^f)$. Since the interest rate is constant in this model, the one year rate is also the instantaneous (overnight) rate.

(b) Find the price/consumption ratio p_t/c_t of the claim to all future consumption in terms of δ, γ, g. Start with

$$p_t = E_t \int_{s=0}^{\infty} e^{-\delta s} \frac{u'(c_{t+s})}{u'(c_t)} c_{t+s} ds,$$

which is the continuos time analogue to

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t} \right)^{-\gamma} c_{t+j}.$$

(You can do this in discrete time if you wish, but the continuos time formula is prettier. In discrete time you will need $\sum_{j=1}^{\infty} z^j = \frac{z}{1-z}$, $\|z\| < 1$. In continuos time, you will need $\int_{s=0}^{\infty} e^{-zs} ds = \frac{1}{z}$, $z > 0$.)

(c) Substitute from part a into part b to express the price/consumption ratio in terms of the interest rate r^f and g. And explain.

(d) Show that if $\gamma < 1$ the price consumption ratio *rises* when consumption growth g increases. Show that if $\gamma > 1$ however, the price/consumption ratio actually *declines* when consumption growth g rises, and show that the price-consumption ratio is the same for all g if $\gamma = 1$ ($u(c) = \ln(c)$.)

(e) How can good news of higher consumption growth g possibly *lower* stock prices? Explain, using parts a, b, and c, and thinking of "cashflow" and "discount rate" (interest rates, here) or "income" and "substitution" effects.

(The stock market often falls on good economic news. This is a model that can help us to understand this effect. It also emphasizes the danger of the usual practice of assuming that cashflow and discount rate news are separate. In this model news about g affects both together, as changes in consumption had both "cashflow" and "discount rate" effects in the habit problem.)

(f) Show that if the investor has log utility $u(c) = \log(c)$, then the ratio of price to consumption is a *constant*, for *any* path for of future consumption (i.e. not just $c_{t+1}/c_t = e^{gt}$), and even if the consumption path is random! Start with $\frac{p_t}{c_t} = E_t \int_{s=0}^{\infty} e^{-\delta s} \frac{u'(c_{t+s})}{u'(c_t)} \frac{c_{t+s}}{c_t} ds/$ Find the value of the constant if $\delta = 0.05$ (a 5% discount rate). (It may be easier to start by looking at the discrete time version, but try to do the continuous time version too.)

(g) What happens if $r^f = g$? What happens if $r^f < g$? Are these values possible?