Problem Set 6 answers

Part I

Carhart:

The introduction summarizes his conclusions:

- Momentum in *stocks* accounts for momentum in *funds*. Funds that did well last year have stocks that went up and those stocks will keep rising a bit. It is *not p*ersistent skill, or good returns for momentum funds. Momentum funds do poorly after transactions costs. There is some persistent *under* performance. Important: Survivor bias free data – includes funds that die. (Lots of hard work by Carhart, and another great CRSP dataset.) (p. 58)

We need to look for the facts! *Find the facts behind these assertions in the paper.*

Now *Questions*

1. Carhart defends the four-factor model as a *performance attribution* model.

   (a) Why is it OK to use a "momentum factor" even if that is not a "state variable for investment opportunities?"

   (b) What *question* are we using the multifactor model to answer, and how is that question different from Fama and French's question?

   A: To measure *stock picking ability*, performance relative to mechanical portfolios is enough, whether or not it is a "true" multifactor model. What you want to know is whether you can replicate the fund's performance with (cheap) index or mechanical strategies, *not* whether the returns from such a strategy or "style" are justified from fundamental risk. Thus, Carhart's model is market, sml, hml, *momentum*, and we don't argue about whether momentum is a "real" factor or not. Anything that you could (and might have!) realistically programmed a computer to do on the right hand side goes here.

   Even better see Fama and French p.1918 pp2 prose.

2. (Hint: Table III is the most important. Spend most of your time to understand it.) How does Carhart form portfolios of mutual funds - -what are Portfolio 1A 1B...10C in column 1 of Table III?

   A: Re-formed once per year based on the previous year's performance.. Then look at monthly returns in the portfolios. Again, we have to form portfolios, then watch. *Keep in mind, all of Carhart's evidence is about* **portfolios** *of funds* (or average behavior of funds of a given type), *not individual funds.*

3. Do the funds that went up last year always continue to go up? How much risk is there in this investment strategy? To quantify these questions, what is the chance that portfolio 1A will earn a positive return in a typical month of next year?

   A: For risk, std dev column. 1A is 0.75% with $\sigma = 5.45\%$. Thus, the chance of going up at all is $\Phi(0.75/5.45) = 55.47\%$. So this is still a 55 up / 45 down phenomenon, and that is among a group of funds in the highest 1/10 of performance for the previous year!

And that's for the *portfolio* of funds – individual funds in the portfolio are even more volatile, so the chance of an individual winner fund doing well next year is even lower. Just a reminder that returns have a lot of risk with them! *Don't confuse alpha with arbitrage opportunity. Alpha means an average return, but not a good return every year.*

Also, as with momentum, this is not a "new phenomoneon" it's a "new way of looking at something we knew all along." There is no conflict here with the conventional wisdom that funds who won last year are almost 50% likely to fall this year. If you can wrap your mind around that, you will have really understood the first column!

4. How do the CAPM R^2 values compare to those for stocks you have seen before? What accounts for the difference?

   A: Especially in the middle, they're huge. Individual companies get about 40%; FF stock portfolios got about 65%, here we're seeing 98%. *Portfolios of active funds are almost exactly replicating the market index.* Note the R^2 tail off as we go up and down the table. *To be extreme, you have to stray far from the benchmark.* The 1 and 10 portfolios actually have very low R^2 meaning huge tracking errors, for funds.

5. Are all the alphas zero after the 4 factor model is done, or is there a puzzle? Who seems still to be outperforming and who is underperforming?

   A: Alphas from -0.1% (1% / year, as before) to -0.2% (9). Then a big increase in the bottom end to an amazing $\alpha = -0.64\%$! Puzzle – how can you lose money in a diversified portfolio?? Efficient markets mean you can't do this either (or you and I short what they are long). Expenses? Coming up.

   Note: negative alphas is not that surprising. The indices do not include transactions costs. Real-world performance is usually less than the index.

   Note: FF4F have no transactions costs or short costs (hml is perpetually short small growth stocks, without any cost), and assume you buy at midpoint of b/a. Some negative alpha is natural.

6. Fund managers claim that fees and turnover do not reduce returns to investors. How could charging more money *not* reduce returns to investors? (Try to be a good salesperson for a high-turnover high-fee fund. Why should I give you my money? Then try to be a good supply-demand economist. What should the equilibrium relationship be between fees, expenses and returns to investors?)

   A: the claim is that fees pay for superior ability, and that turnover is losing dogs and buying good stocks so helps. Standard S=D economics says we should see zero effect. It is a surprise to see any effect! .

7. (Table V. Make sure you understand how this table was created. How are Table IV and Table V different?) What does Carhart find about fees and turnover? How much does a 1% change in fees change returns to investors? How much does turnover – selling one stock and buying another – change returns to investors?

   A: Table V. and p. 67. Table V is based on individual funds, while Table IV looks at the portfolios of Table 1. Expenses and turnover are all bad. Expenses are more than 1-1 bad. Turnover corresponds to a 0.95bp/transaction cost. (Seems large; more than 1-1 as with expenses). I think this is a big puzzle, we should see zeros here in a market in equilibrium! You can't say "investors are dumb" in mutual funds but "investors are smart" in trading!

8. What is the point of Figure 2? (Hint: what would it look like if the sort on one year performance indicated skill?)

A: Skill shouldn't disappear, so the fact that returns seem to revert (2 year return is worse than 1 year return) suggests momentum in stocks (which does dissipate over time). If it were skill, the lines would be flat. This is the crucial evidence that it's momentum in underlying stocks, not skill. Also p. 71, 80% turnover in top funds each year.

9. What does Carhart say about momentum funds – funds that seem to follow a momentum strategy, as revealed by high loadings on the momentum factor? How do we know that we're not seeing the performance of momentum funds? (Hint: no table, but text on p. 73)

   A: p.73. Momentum funds – sorted by PR1YR loadings – do not earn higher returns, let alone four-factor alphas. They lose it all in transactions costs. Hence, chance holders of the winners get a little boost. *This is his evidence that it's momentum in stocks, not momentum funds.* This also raises a strong suspicion about how easy it is to earn momentum returns in practice, once you account for trading costs.

10. *One* year lagged returns are probably mostly luck, not skill. What if you sort funds by the more common 5 year performance averages? (Hint: Figure 3)

    A: Figure 3. The initial expected return spread declines! 5 years is *less* revealing than one year! Other components stay the same. There is *less* effect in the 5 year sort.

Fama and French Mutual Fund Performance

So far, we have been looking for "skill" by guessing some characteristic associated with skill – past returns, MBA by manager, etc. – and looking at the return of a sorted portfolio going forward. This paper tells us whether there is any skill at all, *without* us taking a stand on what characteristic can be used to find good funds. It answers the question "sure the average fund is mediocre, but there are some good funds." Read 1916 top to understand why they're different than persistence tests – if there is skill, lagged returns are a very noisy measure of that skill.

1. What do Fama and French mean by "Equilibrium Accounting? (p. 1915 top)

   A: The average investor must hold the market. Anything else is a zero sum game. (p. 1915)

2. Fama and French focus on the alpha t statistic. Why not look at alphas or information ratios?

   A: Short lived funds are more likely to deliver big alphas. (1924)

3. Explain the numbers in Table 3.

   (a) What does the 95 row, first two columns (95 1.68 1.54) mean? (Hint: At what number x is the probability that a $N(0, 1)$ is larger than x is 5%?)

   (b) Why is the probability of a t greater than 2 or less than -2 not the usual 5% value that we expect for a t statistic?

   A: Table 3 shows you how many funds would have (say) 2.0 alpha t statistic if there really were no alpha but some got lucky. We don't know the degrees of freedom, but for large T t becomes normal, and the 5% probability point for a normal is 1.64. Thus, the "sim" 1.68 tells us that the actual distribution of alpha t statistics, is just a little bit wider than normal. 1.68 means "if there were no alpha then 5% of funds would have alpha t statistics greater than 1.68" 1.54 means "5% of the actual funds have alpha t statistics greater than 1.54" which is *less* than what should happen due to chance if there were any skill. Overall, that's very much what we see. *If there were "good" and "bad" funds, the distribution of alphas would be much more spread out than the simulated distribution* For example, if half the funds were truly +5% and half truly -5%, then we would see a distribution with two humps, and a lot more funds with t>2 or $t < -2$ than the roughly 5% in these tables. See also the figures which make the point visually.

4. Why can't we explain fat tails of estimated alphas by fat tails of the return distribution?

   A: That's the point of using the simulation / bootstrap rather than a t statistic. Large results from big outliers are repeated in the sampling experiments.

5. Do funds look better using only the CAPM in Table AI? IF so, what to FF say about it?

   A: Yes, substantially. There are more good – and more bad – funds. The answer is, the good ones loaded on value (hml) and the bad ones on growth, since smb generated no premium and the funds weren't loading on momentum. Value is alpha to the capm, and there is a spread in value loadings across funds. The average fund does not load on value, but there are value and growth funds.

   A (!) This leads to an obvious answer to the question "why are there good and bad funds" in the main table, especially "why are there bad funds?" There are additional factors such as industry. It is likely that the "good" and "bad" funds in the main tables were just lucky with respect to additional factor bets. That view helps me to understand what negative alpha – before fees – means.

Berk

1. What happens to future returns and flows, according to Berk, if a manager does have some skill?

   A: Berk's central idea: If someone has skill, he can earn better returns. Investors flow in, but all trading strategies have limited scale. The first investors make some alpha, but soon the alpha is driven down to zero. All that happens is that the fund has gotten bigger.

2. Berk says, unlike FF, that managers do have some skill even though alphas are all zero. How can that be?

   A: In the equilibrium above, skill implies larger funds, but no alpha.

3. Berk says that when investors chase past returns, investing in funds that have done well in the past, they are not being irrational, even though future returns are no better than average. How can this be?

   A: Past returns indicate skill. Then investors flow in as above. This drives the future returns back to normal. But it's not irrational.

4. Berk says that even though skill is permanent, returns will not be persistent. Why not?

   A: Same story. The "true" skill is persistent, but actual returns get eaten up.

Part II

1.

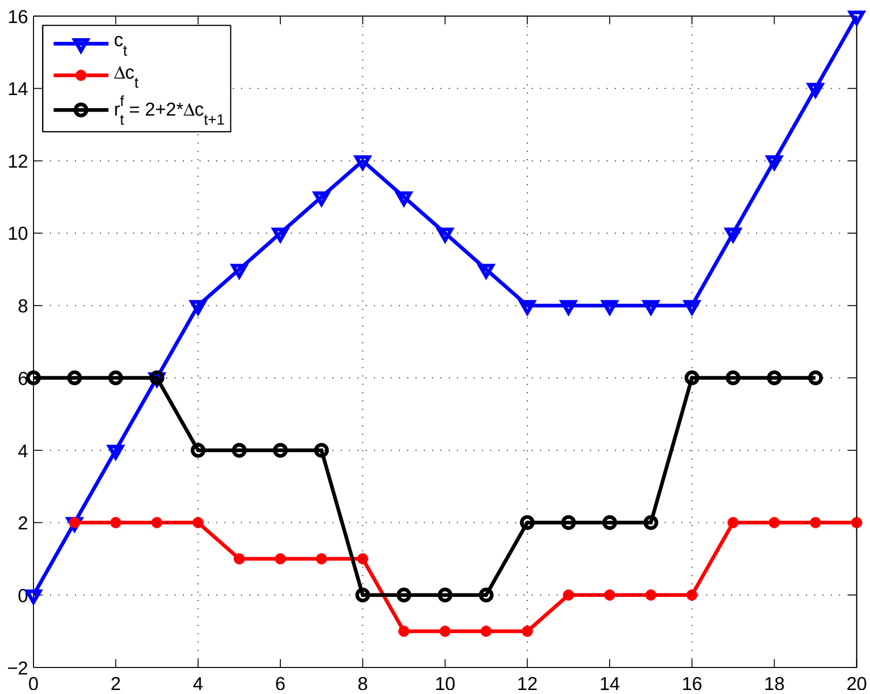
(a)

$$r_t^f = \delta + \gamma E\Delta c_{t+1} = 2 + 2 * E\Delta c_{t+1}$$

$E\Delta c$	-1	0	1	2
$2 + 2 * E\Delta c$	0	2	4	6

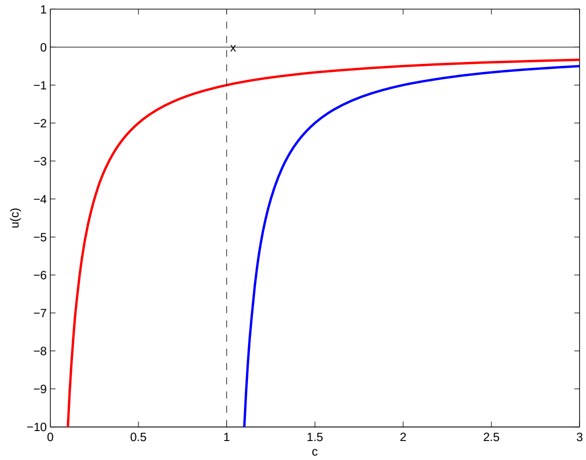
I graphed Δc_{t+1} in red and r_t^f in black. r_t^f is the interest rate quoted at time t for loans from t to $t+1$, and is conventionally dated as of time t since it is observed at time t. I graphed it that way. The hard part is the t vs t+1. The interest rate at t reflects consumption growth over the *next* year.

(b) You see very interesting patterns here. First, note that the interest rate moves one period before the peaks of the consumption series. It looks like the interest rate change is "causing" the consumption change, though nothing of the sort is happening. Asset prices are forward looking, so interest rates "cause" consumption changes the same way weather forecasts "cause" the weather. Second, note that interest rates (black) correlate with consumption growth rates. Measured by growth, interest rates are low in recessions. Third, growth and interest rates are low when levels are declining. Interest rates move ahead of recessions as defined by the *level* of consumption. Much popular discussion confuses the level and growth views of where we are in economic cycles. Much popular discussion attributes all of the correlation to the Fed's control of interest rates, but that is completely absent here. There is no Fed in this model.



2.

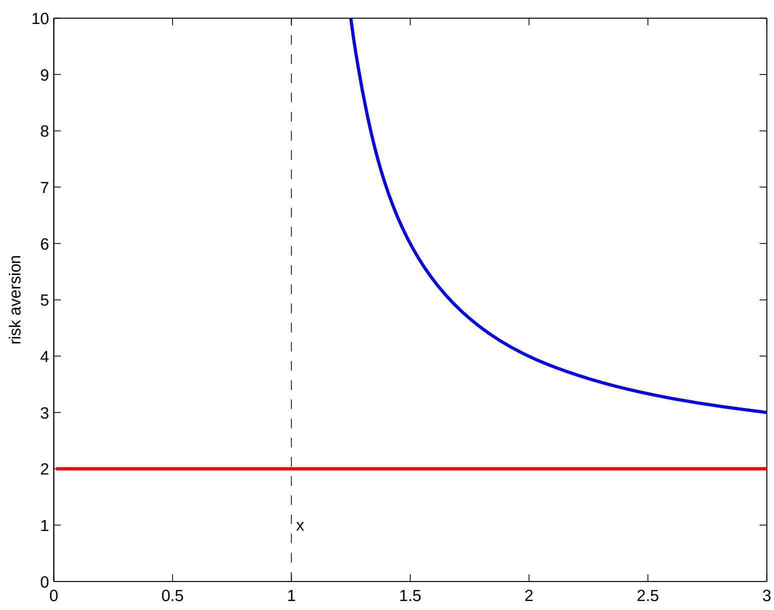
(a)



X now looks like zero

(b) The derivation.

$$
\begin{aligned}
u(c) &= \frac{(c-X)^{1-\gamma}}{1-\gamma} \\
u'(c) &= (c-X)^{-\gamma} \\
u"(c) &= -\gamma(c-X)^{-\gamma-1} \\
\eta &\equiv -\frac{cu''(c)}{u'(c)} = \frac{c\gamma(c-X)^{-\gamma-1}}{(c-X)^{-\gamma}} \\
\eta &= \gamma\frac{c}{c-X}
\end{aligned}
$$



So, risk aversion rises as consumption c falls relative to the habit, or minimum level where you can't make the mortgage payments, X. *Relative* risk aversion rises because consumption losses that are a huge fraction of $c-X$ – how far you are above disaster – are a small fraction of c – how far you are above zero, and thus how large a risk is relative to invested wealth.

(c)

$$
\begin{aligned}
E(R^e_{t+1}) &= \eta_t cov_t(R^e_{t+1}\Delta c_{t+1}) \\
E(R^e_{t+1}) &= \gamma \frac{c_t}{c_t - X}\sigma^2_{\Delta c}
\end{aligned}
$$

The expected excess return rises as consumption falls.

(d)

$$
\frac{P_t}{c_t} = \frac{E_t(c_{t+1}/c_t)}{E_t(R^e_{t+1})} = \frac{c_t - X}{c_t}\sigma^2_{\Delta c_t}
$$

Prices fall as consumption falls, generating the rise in expected returns.

(e) In the situation, $\frac{c_t - X}{c_t}$ fell from 0.2 to 0.1, so the price-consumption ratio falls 50%! Prices fall by

$$
\begin{aligned}
\frac{P_t}{P_{t-1}} &= \frac{P_t/c_t}{P_{t-1}/c_{t-1}}\frac{c_t}{c_{t-1}} \\
\frac{P_t}{P_{t-1}} &= 0.50 \times 0.90 \approx 0.45
\end{aligned}
$$

55%! (if you just added the 10% consumption growth and said 60% that's fine.) The point – a lot more than 10%. You get amplification of shocks, huge. The reason is, *we add a huge expected return shock to the small cashflow shock.*

(f)

$$
\begin{aligned}
m_{t+1} &= \beta \frac{(c_{t+1} - X)^{-\gamma}}{(c_t - X)^{-\gamma}} \\
&= \beta \frac{(c_{t+1}\left(1 - \frac{X}{c_{t+1}}\right))^{-\gamma}}{\left(c_t\left(1 - \frac{X}{c_t}\right)\right)^{-\gamma}} \\
&= \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{\frac{c_{t+1}-X}{c_{t+1}}}{\frac{c_t - X}{c_t}}\right)^{-\gamma} \\
\log(m_{t+1}) &= -\delta - \gamma \log\left(\frac{c_{t+1}}{c_t}\right) - \gamma \log\left(\frac{\frac{c_{t+1}-X}{c_{t+1}}}{\frac{c_t - X}{c_t}}\right)
\end{aligned}
$$

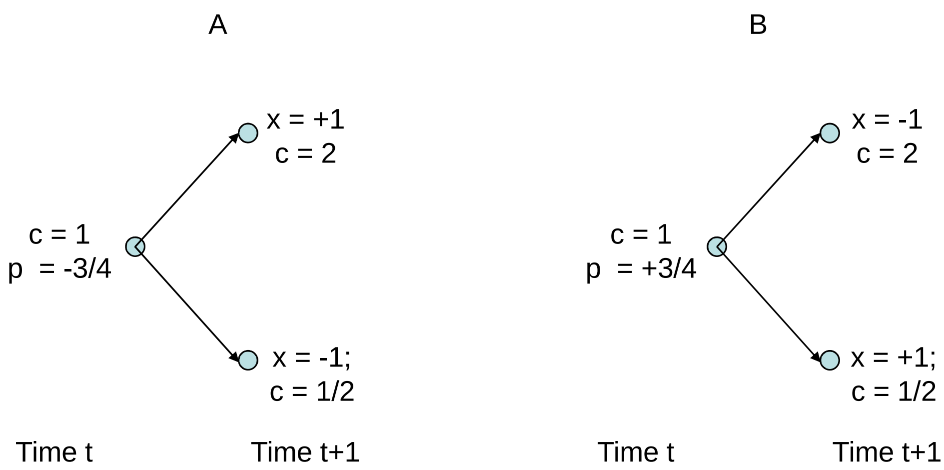
(g)

$$
\log(m_{t+1}) = a - \gamma \log\left(\frac{c_{t+1}}{c_t}\right) - \gamma \log\left(\frac{\frac{p_{t+1}}{c_{t+1}}}{\frac{p_t}{c_t}}\right)
$$

3. For any payoff $x = \{x_u, x_d\}$,

$$
p = E(mx) = \pi_u \frac{1}{c_u}x_u + \pi_d \frac{1}{c_d}x_d
$$

$$
p = E(mx) = \frac{1}{2}\frac{1}{2}x_u + \frac{1}{2}\frac{2}{1}x_d = \frac{1}{4}x_u + x_d
$$



(a) For the bond, $x_u = x_d = 1$

$$p = E(mx) = \frac{1}{4}1 + 1 = 1.25$$

A bond price can exceed one meaning a negative real interest rate. 1/2 is such a terrible outcome that the consumer would really like to save to prevent it.

(b)

$$p = \frac{1}{4}1 + (-1) = -0.75$$

The price is negative. Well, losing a dollar in the state of the world that consumption goes down by half is a terrible idea, and you would pay not to take that bet.

(c)

$$p = \frac{1}{4}(-1) + 1 = 0.75$$

The situation is exactly reversed

(d) The mean $E(x) = 0$ is the same and the variance is the same. They differ by *in which state of nature* you take losses. That matters *This is important. Risk is not standard deviation, its covariance with consumption.*

(e)

$$\begin{aligned} p &= \frac{1}{4}1 + 0 = \frac{1}{4} \\ p &= \frac{1}{4}0 + 1 = 1 \end{aligned}$$

The claim that pays in the bad state is much more valuable. You're hungrier in the bad state and willing to pay more

(f) Part b is +1 contingent claim to the good state and -1 contingent claim to the bad state. The value of this arbitrage portfolio is $1/4 - 1 = -3/4$, the same value..

4.

(a)

$$\begin{cases} S = \frac{1}{2}m_u u\ S + \frac{1}{2}m_d d\ S \\ 1 = \frac{1}{2}m_u R^f + \frac{1}{2}m_d R^f \end{cases}$$

$$\begin{cases} 2 = m_u u + m_d d \\ 2 = m_u R^f + m_d R^f \end{cases}$$

$$
\begin{aligned}
\begin{bmatrix} u & d \\ R^f & R^f \end{bmatrix} \begin{bmatrix} m_u \\ m_d \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
\begin{bmatrix} m_u \\ m_d \end{bmatrix} &= \begin{bmatrix} u & d \\ R^f & R^f \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
\begin{bmatrix} m_u \\ m_d \end{bmatrix} &= \frac{1}{u-d} \begin{bmatrix} 1 & -\frac{d}{R^f} \\ -1 & \frac{u}{R^f} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
\begin{bmatrix} m_u \\ m_d \end{bmatrix} &= \frac{2}{u-d} \begin{bmatrix} 1-\frac{d}{R^f} \\ -1+\frac{u}{R^f} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
m_u &= 2\frac{1}{R^f}\left(\frac{R^f-d}{u-d}\right) \\
m_d &= 2\frac{1}{R^f}\left(\frac{u-R^f}{u-d}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
C &= E(m \times \max(S_1 - S, 0)) \\
&= \frac{1}{2} m_u \times (u\ S - S) + \frac{1}{2} m_d \times 0 \\
&= \frac{1}{R^f}\left(\frac{R^f-d}{u-d}\right)(u-1)\,S \\
&= \left(\frac{R^f-d}{R^f}\right)\frac{u-1}{u-d}S
\end{aligned}
$$

(c)

$$
\begin{cases} k\ u\ S + hR^f = u\ S - S \\ k\ d\ S + hR^f = 0 \end{cases}
$$

$$
\begin{cases} k\ u\ + hR^f/S = (u-1) \\ k\ d\ + hR^f/S = 0 \end{cases}
$$

$$
\begin{bmatrix} u & R^f/S \\ d & R^f/S \end{bmatrix} \begin{bmatrix} k \\ h \end{bmatrix} = \begin{bmatrix} u-1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} k \\ h \end{bmatrix} = \begin{bmatrix} u & R^f/S \\ d & R^f/S \end{bmatrix}^{-1} \begin{bmatrix} u-1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} k \\ h \end{bmatrix} = \frac{1}{u-d} \begin{bmatrix} 1 & -1 \\ -d\frac{S}{R^f} & u\frac{S}{R^f} \end{bmatrix} \begin{bmatrix} u-1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} k \\ h \end{bmatrix} = \frac{u-1}{u-d} \begin{bmatrix} 1 \\ -d\frac{S}{R^f} \end{bmatrix}
$$

the value is then

$$
\begin{aligned}
kS + h &= \frac{u-1}{u-d}S - d\frac{S}{R^f}\frac{u-1}{u-d} \\
&= \frac{u-1}{u-d}S\left(1 - \frac{d}{R^f}\right) \\
&= \frac{u-1}{u-d}S\left(\frac{R^f-d}{R^f}\right)
\end{aligned}
$$

Yes, it's the same!

(d)

$$\begin{aligned} S_0 &= \frac{1}{R^f}\left[(\pi_u^* u S_0) + (1-\pi_u^*)(dS_0)\right] \\ R^f &= \pi_u^* u + (1-\pi_u^*)d \\ \pi_u^* &= \frac{R^f - d}{u-d} \\ \pi_d^* &= 1 - \frac{R^f-d}{u-d} = \frac{R^f-u}{d-u} \end{aligned}$$

$$C_0 = \frac{1}{R^f}\left\{\pi_u^* \max(uS_0 - X, 0) + \pi_d^* \max(dS_0 - X, 0)\right\}$$

for $X = S_0$

$$\begin{aligned} C_0 &= \frac{1}{R^f}\left\{\pi_u^* (u-1)\right\} \\ &= \frac{1}{R^f}\frac{R^f-d}{u-d}(u-1) \end{aligned}$$

This is (of course!) the same answer.

(e)

$$\pi_u^* = \frac{R^f-d}{u-d}; \pi_d^* = \frac{R^f-u}{d-u}$$

$$m_u = 2\frac{1}{R^f}\left(\frac{R^f-d}{u-d}\right); m_d = 2\frac{1}{R^f}\left(\frac{u-R^f}{u-d}\right)$$

Hmm. It looks like

$$\begin{aligned} \pi_u^* &= R^f \pi_u m_u = \frac{\pi_u m_u}{\pi_u m_u + \pi_d m_d} \\ \pi_d^* &= R^f \pi_d m_d = \frac{\pi_d m_d}{\pi_u m_u + \pi_d m_d} \end{aligned}$$

or, more deeply with

$$\pi_u^* = \pi_u R^f \beta \frac{u'(c_u)}{u'(c)} = \pi_u \frac{\beta\frac{u'(c_u)}{u'(c)}}{\beta\frac{u'(c_u)}{u'(c)} + \beta\frac{u'(c_d)}{u'(c)}} = \pi_u \frac{u'(c_u)}{u'(c_u)+u'(c_d)}$$

Risk neutral probabilities are actual probabilities multiplied by marginal utilities. Risk aversion is the same thing as over-wegithing the probabilities of unpleasant states.

This is a very deep theorem. People who pay too much attention to un pleasant states are not being irrational, they're just being risk averse. In fact, overweighting probabilities of bad states is a good way to think of and implement risk-averse plans. The whole "behavioral" – people are dumb they get probabilities wrong – vs. "rational" – people are smart, they know probabilities but they are risk averse – is *completely meaningless* unless you can tie down $u'(c)$. That requirement holds for both sides of the debate.

This theorem has huge practical implications as well. *Everything* that you do by "risk neutral pricing" can be done by "discount factors" and vice versa. Choose the representation that makes the calculations easier.

(f) Recall

$$\begin{aligned} \pi_u^* &= \frac{R^f-d}{u-d} \\ \pi_d^* &= \frac{R^f-u}{d-u} \end{aligned}$$

$\pi_u^* < 0$ and $\pi_d^* > 0$ if $R^f < d$. That means that the stock gives a better rate of return in *both* states of nature. Stocks strictly dominate bonds – there is an arbitrage opportunity. Borrow, invest in stocks, and *no matter what happens* you make money. Similarly, if $R^f > u$ then $\pi_d^* < 0$ and $\pi_u^* > 0$. In this case, the bond does better than the stock *no matter what happens*, so sorting the stock is an arbitrage. So, really, we have to state the problem with the additional assumption that $d < R^f < u$, or "and there are no arbitrage opportunities."

You have just discovered one of the most basic theorems of the theory of finance: *"There exists a positive discount factor $m_u > 0$, $m_d > 0$ and a set of valid risk-neutral probabilities $0 < \pi_u^* < 1, 0 < \pi_d^* < 1$ if and only if there are no arbitrage opportunities.*" (The precise version handles $\geq$ vs. $>$.)

5. Risk sharing

   (a) We already found m, and there wasn't any free choices. So, $m^A = m^B = m$, ex-post.

   (b)

$$\begin{aligned}\left(\frac{c_{t+1}^A}{c_t^A}\right)^{-\gamma_A} &= \left(\frac{c_{t+1}^B}{c_t^B}\right)^{-\gamma_B} \\ \gamma_A \log\left(\frac{c_{t+1}^A}{c_t^A}\right) &= \gamma_B \log\left(\frac{c_{t+1}^B}{c_t^B}\right)\end{aligned}$$

   and with the given numbers

$$\log\left(\frac{c_{t+1}^A}{c_t^A}\right) = 2\log\left(\frac{c_{t+1}^B}{c_t^B}\right)$$

   consumption of the two investors move in lockstep. The first investor has consumption twice the second, so both mean and standard deviation are twice as big. Mr. A is less risk averse, he takes greater risk but gets greater reward.

   (c) Income doesn't enter the problem. They use asset markets to achieve perfect insurance, and share all risks equally. Big picture *asset markets exist to share risks!*

   Real world asset markets do not completely share risks in this way. Our consumptions do not move in lockstep regardless of our incomes. That is because real asset markets are not "complete." There are more states of nature than securities, so we can't buy or sell contingent claims on every outcome, and our individual fortunes in labor markets in particular. However, the general version of this theorem is that people optimally share all the tradeable risks, so asset markets achieve as much risk sharing as is possible given their span (how many states of nature are traded.) In equations, $proj(m^i|X) = proj(m^j|X)$ generalizes $m^i = m^j$ where X is the set of traded payoffs. This is explained in *Asset Pricing.*

6.

   (a)

$$R^f = 1/E(m) = \frac{1}{\beta}\left(\frac{c_t}{c_{t+1}}\right)^{-\gamma} = G^\gamma/\beta$$

$$\begin{aligned}R^f &= 1/E(m) = e^\delta e^{\gamma g} \\ r^f &= \delta + \gamma g\end{aligned}$$

   As before, interest rates are higher if consumption growth is higher. For $\gamma = 1$, the interest rate moves one for one with consumption growth. If γ is larger, interest rates move more than consumption growth.

(b)

$$\begin{aligned} p_t &= \int_{s=0}^{\infty} e^{-\delta s}\left(\frac{c_{t+s}}{c_t}\right)^{-\gamma} c_{t+s}ds, \quad c_{t+s}=c_t e^{gs} \\ \frac{p_t}{c_t} &= \int_{s=0}^{\infty} e^{-\delta s}\left(e^{gs}\right)^{1-\gamma} ds = \int_{s=0}^{\infty} e^{-[\delta-(1-\gamma)g]s}ds = \frac{1}{\delta-(1-\gamma)g} \end{aligned}$$

In discrete time,

$$\frac{p_t}{c_t} = \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{1-\gamma}$$

$$\frac{p_t}{c_t} = \sum_{j=1}^{\infty} \beta^j G^{(1-\gamma)j} = \frac{\beta G^{1-\gamma}}{1-\beta G^{1-\gamma}} = \frac{e^{-\delta}e^{(1-\gamma)g}}{1-e^{-\delta}e^{(1-\gamma)g}}$$

it's approximately the same, but not as pretty.

(c)

$$\frac{p_t}{c_t} = \frac{1}{r^f - g}$$

This is the famous "gordon growth" formula. Price is higher if growth is higher or discount rate is smaller.

(d)

$$\frac{p_t}{c_t} = \frac{1}{\delta-(1-\gamma)g}$$

(e) If $\gamma > 1$, higher g raises the denominator and lowers prices! Looking at a, higher g raises the interest rate by γg, and looking at c, higher g lowers the cashflow effect by g. If $\gamma > 1$, the interest rate effect outweighs the cashflow effect. In words, if $\gamma > 1$, when you find out you will be richer in the future, you really want that wealth today, so you try to sell stocks, pushing down the price.

The stock market does often fall on good news. Commentators often blame the Fed – the Fed will raise interest rates in response to good economic news. In this model, the interest rates are going up whether the Fed likes it or not, producing the decline in stock prices on good news. More importantly, you see here how the "new view" that risk aversion is pretty high means that *discount rate* effects can make stock prices pretty hard to understand.

(f)

$$\frac{p_t}{c_t} = E_t \int_{s=0}^{\infty} e^{-\delta s}\left(\frac{c_{t+s}}{c_t}\right)^{-\gamma} \frac{c_{t+s}}{c_t}ds$$

$$\frac{p_t}{c_t} = \int_{s=0}^{\infty} e^{-\delta s}ds = \frac{1}{\delta}.$$

$$\frac{1}{0.05} = 20$$

So the $\gamma = 1$ case holds no matter what consumption does! Each of the little income vs. substitution effects always cancel.

(g) In these cases, prices are infinite.

Infinite prices actually are relevant, and capture some important policy issues. For example, if the US economy and tax receipts grow faster than the real interest rate, then the present value of taxes is infinite. Schemes like social security can in fact manufacture money out of thin air; they are a way of exploiting the infinite present value of government revenues. OTOH, if the interest rate is higher than the growth rate of the economy, you and I will have to pay for social security's largesse.