$r < g$

John H. Cochrane*

July 28, 2021

Abstract

A situation in which the rate of return on government bonds r is less than the economy's growth rate g suggests that borrowing has no fiscal cost. I argue instead that $r < g$ is irrelevant for the current US fiscal problems. $r < g$ cannot begin to finance current and projected deficits. $r < g$ does not resolve exponentially growing debt. $r < g$ can finance small deficits, but large deficits still need to be repaid by subsequent surpluses. The appearance of explosive present values comes by using perfect-certainty discount formulas with returns drawn from an uncertain world. Present values can be well behaved despite $r < g$. The $r < g$ opportunity is like the classic strategy of writing put options, which fails in the most painful state of the world.

*Hoover Institution, Stanford University and NBER. https://www.johnhcochrane.com. This essay is based on comments on Reis (2021) delivered at the February 19, 2021 NBER Economic Fluctuations and Growth conference.

1 Introduction

Fiscal sustainability is the most important macroeconomic issue of our time. The US is embarked on a historically unprecedented peacetime fiscal expansion. The debt to GDP ratio, passing 100%, is already higher than it has ever been. And current deficits, spending plans, and looming entitlements mean we are only halfway done. Will this work out?

$r < g$ seems to offer a delicious opportunity. Briefly, suppose the government borrows a huge amount, and simply rolls over the debt, borrowing new money to pay principal and interest on the old. Then debt grows at the rate of return on government debt, r. But if GDP grows at a greater rate $r < g$, then the ratio of debt to GDP slowly declines. The borrowing need never be repaid by higher later tax revenues or lower spending. Public debt, apparently, has no fiscal cost.

Washington understands the logical implications of the proposition that debt has no fiscal cost better than economists, and Washington is acting on it. We are citizens of a democracy. This is our government. If our government can borrow, and never worry about paying back debts, why should any of us pay back debts? Why should the government not borrow, and repay our student debts, mortgage debts, business debts; bail out state and local pension promises. Why should we pay taxes? Why should we work? Why should our government not just send us money and we can all stay home and order stuff from Amazon?

The federal government is more and more just borrowing and sending checks to people, businesses, and other levels of government. Indeed, since the Fed is buying most treasury debt, the government is basically printing money and sending it to people. The sense that such government spending, must eventually come from taxation has totally disappeared from policy discourse.

Will this work? Does a new economic configuration, or new theoretical understanding, mean that a vast free store of national wealth is sitting there untapped? Or, will this unprecedented fiscal expansion lead to inflation that will make the 1970s seem like a walk in the park, a severe fiscal reckoning, a catastrophic US sovereign debt crisis, or an inability to borrow in the next great financial, health, or military crisis, and thereby far worsen that crisis?

Six trillion dollars ago, Olivier Blanchard delivered his American Economic Association Presidential Address, Blanchard (2019), analyzing debt sustainability in the $r < g$ framework. I believe Blanchard's fiscal policy address will be seen as important in our time as Milton Friedman's monetary policy address Friedman (1968) was in his. Each analyzed a dramatic experiment just as the government got seriously going on it, with clear simple and innovative economics. Blanchard brings a career's worth of thought and mastery of a large and difficult literature to the question. Friedman offered a Babe-Ruth-worthy called home run: if you try this, it will fall apart and here's why. The government promptly tried it, and it fell apart. Blanchard is more nuanced, but his summary "public debt may have no fiscal cost," and welfare benefits, shows sympathy with the view that it just might work, despite outlining mechanisms why it might not.

2 The usual argument

So does $r < g$ mean that public debt has no fiscal cost? The debt-to-GDP ratio evolves as

$$\frac{d}{dt}\left(\frac{b_t}{y_t}\right) = (r_t - g_t)\frac{b_t}{y_t} - \frac{s_t}{y_t}. \tag{1}$$

with b = real value of debt, y = GDP, r = rate of return, g = GDP growth rate, s = real primary surplus. (Start with $db_t/dt = rb_t - s_t$.) $r < g$ seems to offer a delicious scenario: Run up the debt with a string of big deficits. Then, just keep rolling over the debt without raising surpluses. Debt grows at r, but GDP grows at g, so the debt-to-GDP ratio slowly declines at rate $r - g$. Apparently debt never has to be repaid by higher surpluses. In that sense, debt has "no fiscal cost."

Debt repayment is related to present value relations. If we solve this differential equation forward,

$$\frac{b_t}{y_t} = \int_{\tau=0}^{T} e^{-(r-g)\tau} \ \frac{s_{t+\tau}}{y_{t+\tau}} d\tau + e^{-(r-g)T} \ \frac{b_{t+T}}{y_{t+T}}$$

$r < g$ seems to imply that government debt is infinitely valuable, or that it contains a "bubble" terminal condition that can be "mined."

But this analysis suggests two ridiculous conclusions. First, it seems there are no fiscal limits at all. As above, the government can borrow, send us "stimulus" checks, and nobody has to work again. Well, obviously not.

Second, it seems that a theoretical wall separates $r > g$ from $r < g$. If r is one basis point (0.01%) above g, we solve the differential equation forward to a present value, debts must be repaid, the government must return to fiscal "austerity" to ward off the "bond vigilantes" who might trigger hyperinflation or sovereign default. If r is one basis point below g, we should really solve the integral backward, debts never need to be repaid, the government may borrow and spend, or just give away money to voters, with no repercussions. Well, obviously not.

Why not? The conventional limitation is the fact that $r < g$ eventually cannot scale. Sooner or later more debt raises r. *Marginal* $r - g$ is what counts to fiscal expansion.

Therefore *there is a maximum debt/GDP ratio out there somewhere. The fiscal expansion cannot be unlimited or go on forever.*

This consideration still suggests a fiscal expansion up to the debt/GDP ratio where $r = g$, however. And that limit may be a long way away. There isn't an infinite money tree, but there may be a $30 trillion bill lying on the sidewalk.

For example, standard investment crowding out is one mechanism that raises r if we overdo it. Once the government channels all savings into consumption, there is no savings left to create a capital stock. As capital depreciates, the marginal product of capital and hence the interest rate must rise. But such crowding out seems a long way away, and something we would easily see approaching by a slow rise in real interest rates.

If $r < g$ is driven by low r due to a liquidity premium, or a money-like demand for government debt, that demand declines more swiftly than crowding-out as debt increases, suggesting a much lower limit.

Most salient to me, high debts leave us open to doom-loop run dynamics. If markets sniff a crisis coming, they charge higher rates as a default premium. Higher rates mean higher debt service which explodes the debt faster, and then the default happens. Greece on steroids with no Germany to bail us out. Such an event is most likely to happen in a recession or other crisis, and if political chaos continues, just when fresh borrowing will be most needed. Imagine if the US wants to bail out the financial system again, and markets say no.

The longer we spend with elevated debt/GDP ratios, the longer we are in danger of such an earthquake. Leaving ample unused fiscal space stops doom loops, and might also come in handy in the next unforeseen crisis. It's a good thing that WWII did not *start* with 100% debt to GDP already on the books. (Borrowing long term rather than rolling over short term debt would be immensely beneficial as well.)

I am particularly attracted to the view that government debt is a negative beta security. In recessions, inflation goes down and interest rates go down, so bond prices go up. Bond returns are great, negative beta. Negative beta drives low average returns. The 1970s had the opposite pattern. Clearly, "unanchoring" of inflation expectations could drive us to the 1970s regime and a sharp rise in r.

This is the usual argument, and one's judgement can come down in many ways. The benefits of fiscal expansion, which I have not mentioned, matter too. If you think the government will make wise public investments, your case is stronger. If you think it will at best send checks to voters and at worst waste the money on crony boondoggles, the case is weaker.

3 Beside the point

I want to emphasize a novel, and more radical view of the issue: *The $r < g$ debate is irrelevant to current US fiscal policy issues.* I think economists have to some extent chased a theoretically interesting rabbit down a hole, while the classic and important issues fester.

The main scenario contemplates a "one-time" fiscal expansion, and then run a few decades of zero primary surplus while $r < g$ whittles down the debt/GDP ratio.

An important second scenario notes that $r < g$ allows the government to run a steady primary deficit and keep a constant debt/GDP ratio. In the equation, zero on the left hand side means the two terms on the right cancel. At our 100% debt/GDP, and 1% $r < g$, we can run a steady 1% of GDP primary deficit, \$200 billion today, as long as $r < g$ lasts.

But, as I illustrate in Figure 1 and Figure 2, the US runs \$1 trillion, 3-5% of GDP deficits in good times, and \$5 trillion, 25% of GDP deficits in each decade's once-in-a-century crises. And then in about 10 years unfunded Social Security, Medicare, and other entitlements really kick in.

Zero primary surplus while $r < g$ whittles down the debt/GDP ratio means *zero* surpluses, not perpetual 5% of GDP deficits. Zero primary surplus means taxes that *equal* spending, not taxes = 0. The promise never was no taxes, the promise was no *extra* taxes, on the assumption that taxes equal spending already! A commitment to two generation's

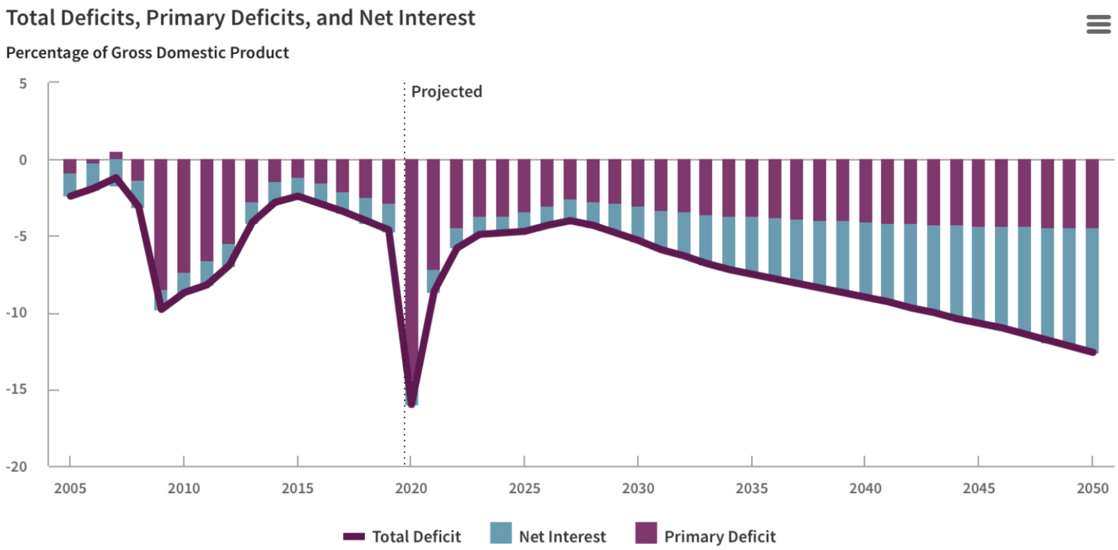

Figure 1: Deficits. Source: Congressional Budget Office https://www.cbo.gov/publication/56516

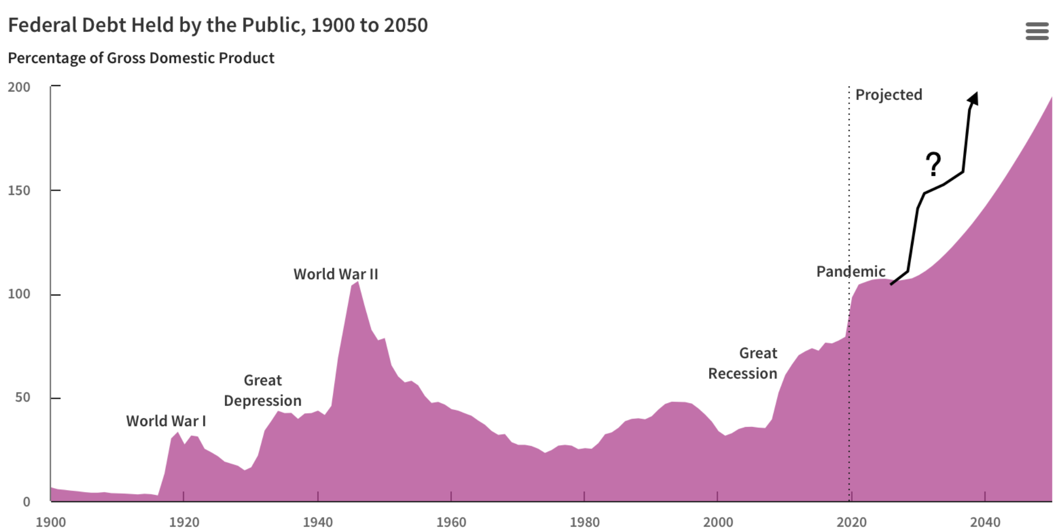

Figure 2: Debt to GDP ratio. Black line: An artistic guess that includes occasional crises. Source: Congressional Budget Office https://www.cbo.gov/publication/56516

worth of zero primary surpluses, steadily whittling down the debt to GDP ratio by one percent per year, would be a conservatives fiscal hawks' dream, and a dramatic fiscal tightening for the US.

The US has *exponentially growing* debt to GDP, not gently declining debt to GDP that can be pushed to decline from a higher level. We do not start a "one-time" fiscal expansion with zero current and planned primary surpluses, as the scenario posits. And the spending plans our government is now proposing are hardly a one-time expansion. New social programs and a century's worth of climate subsidies are a new perpetual deficit, not a one-time expansion.

To the scenario of a steady debt-to-GDP ratio with perpetual deficits, $r - g$ of 1% allows a 1% of GDP steady primary deficit, not 5% in good times, 25% in bad times, and then pay for Social Security and health care, all before we start the fiscal expansion.

The opportunity also has to last a long time. $r < g$ of 1% means that even with a return to zero primary surpluses, the US debt to GDP ratio declines one percent per year. Figure 3 illustrates that path. If the US raises debt from 100% of GDP to 150%, – less than we already have done since 2008 – and $r - g = -1\%$, we need 40 years of taxes actually equal to spending just to bring the debt / GDP ratio back to 100%, and 110 years to reduce debt/GDP to a historically more comfortable 50%. If we go up to 200%, those numbers are 70 years and 139 years. That's a long time to hope the bond vigilantes stay at bay, and we don't have a crisis that demands another "one time" fiscal expansion.

More importantly, these decades force us to think much harder about the source of $r < g$. If the $r < g$ opportunity vanishes while we are still at a high debt to GDP ratio, a sudden fiscal reckoning awaits. There are many stories for $r < g$. The simplest of course is standard growth theory: $r = \delta + \gamma(g - n) = f'(k)$ suggests that lower growth, driven by slower technical change produces a low rate. The $r < g$ literature is full of more fun stories, "savings gluts," a "demand for safe assets," foreign central banks, and so forth. None of these are understood with the clarity one would want to tempt the bond vigilantes for half a century, at least in my judgement. I flag them as important but move on.

Now, larger debt can finance a larger primary deficit. 400% debt/GDP with $r < g$ unchanged at 1%would allow us to finance 4% of GDP forever. So one might combine the two scenarios, run up the debt and then finance larger permanent deficits! However,

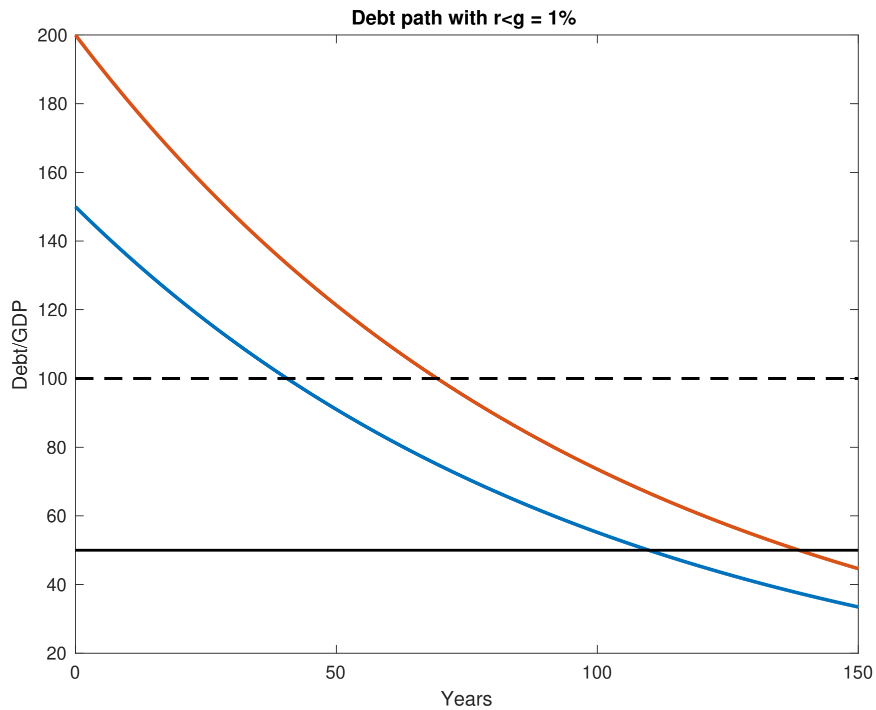

Figure 3: Debt paths with $r < g$ = 1%. Red starts at 200%, blue starts at 150%. Horizontal lines indicate 50% and 100% for reference.

like other sober writers including Blanchard, I posit that 400% debt to GDP forever either raises r, or keeps us too close to the doom loop to be a realistic scenario. So, I'll focus on the current proposals, a "one-time" expansion which we grow out of, getting back to something like normal below 100% debt to GDP, or a permanent deficit with less than 100% debt to GDP.

Looking at flows also makes sense of the apparent $r = g$ discontinuity. As we move from $r - g = 0.01\%$ (1 basis point) to $r - g = -0.01\%$ at 100% debt to GDP, we move from a steady 0.01% of GDP ($2 billion) surplus, to a steady 0.01% ($2 billion) of GDP deficit. That's not going to finance anyone's federal spending wish list! *This* transition is clearly continuous.

The opportunity to grow out of debt with $r - g = -0.01\%$, means a 150% debt to GDP will, with zero primary surpluses, resolve back to 50% debt to GDP in 11,000 years. $(-\log(0.5/1.5)/0.0001 = 11,000.)$ This is not much different than the infinity, and beyond, required by $r > g$. A sensible understanding of how equations map to the economy is continuous as r passes g. If there is a "wealth effect," a transversality condition violation in debt to GDP that grows at 0.01%, rising from 150% by a factor of 3 to 450% in 11,000 years, then there is surely a "wealth effect" in a debt to GDP ratio that takes 11,000 years to decay by a factor of 3 from 150% to 50%.

This is a quantitative question. $r < g$ of 10% would solve our problems. (Or invite appetites for larger problems!) But $r < g$ of 1% is a factor of 5 at least too small to make a dent in the US current fiscal situation. $r < g$ of 1% would solve a 1% problem. Our problem is at least a factor of 5 larger.

So what does $r < g$ mean? $r < g$ may shift the *average* surplus to a slight perpetual deficit, just as seigniorage allows a slight perpetual deficit. But any substantial *variation* in deficits about that average – business cycles, wars, infrastructure programs – must be met by a substantial period of above average surpluses, to bring back debt to GDP in a reasonable time. And $r < g$ is by any reasonable measure far too small to make a dent in the US *existing* fiscal problem, let alone a "one-time" fiscal expansion that is really a multi-decade increase in permanent deficits.

What about the post WWII experience? Does that not prove $r < g$ works? In some ways yes, and in those ways the experience points to all our dangers today. Until about 1975, the US ran steady small primary surpluses, s, with small deficits in recessions balanced by larger surpluses in expansions. There were no perpetual deficits, and there were no looming entitlements and pensions. The US experienced supply-side growth g, driven by productivity, in a much less regulated economy, faster than in all history, and faster than anyone expected in 1945. There was widespread financial repression holding down r, regulations limiting interest on bank accounts, and capital controls. Few people held stocks. WWII was financed with long-term debt, keeping a doom loop at bay. We have the opposite of all these circumstances today. And with all that, we still had an inflation in the late 1940s, cumulatively raising the price level 40% and wiping out roughly that much of the debt in real terms; we had a currency crisis in 1971 leading to the abandonment of Bretton Woods and devaluation of the dollar, and we had the inflation of the 1970s.

What about Japan? Japan has 250% debt to GDP and very low interest rates. For now. Japan sits on many earthquake faults, and this is one. Japanese debt however is largely long term, held either by Japanese people or the Japanese central bank; Japan has assets accumulated by years of trade surpluses, and Japan has an estate tax. Part of the US "privilege" is our debt is held by foreign central banks and both domestic and foreign financial institutions, welcome but very hot money sources of funding. Our debt is short term, and we have been running trade deficits. Just because a bubble has run for a long time, just because the fault has not ruptured in a few decades, does not repeal the economic danger that the fault represents. Moreover, if one makes a list of countries with high debt to GDP ratios, Japan is about the only one that one would think a pleasant possibility.

4 Which r?

But, as the University of Chicago saying goes, enough of the real world, how does $r < g$ work in theory? How do we reconcile the above simple analysis with present value formulas, where $r < g$ even of one basis point seems to offer manna from heaven? How do we think of debt as the present value of surpluses if surpluses can be perpetually negative? Which r should we use? Average returns on US government debt are low. But average returns on equity, and measures of the marginal product of capital are comfortably higher than g.

Bottom line: Correctly-calculated present values may converge, indicating that debts must be paid, even when r, as measured by the rate of return on government debt, is below g, the average growth rate of the economy. This is most recently a major point of Reis (2021).

In a frictionless world of perfect certainty all interest rates are the same. That we have a choice tells us that the $r < g$ that we measure comes from a world with uncertainty and potentially liquidity premiums.

But it is misleading to pluck one measure, generated from our world, and use it in a perfect-foresight present value formulas. Our world can produce rates of return that, put in perfect foresight formulas, generate false infinities and false manna from heaven. Present value formulas that correctly reflect the uncertainty or liquidity that generates multiple r options still give the right answer, a finite value of debt indicating that debts must be repaid, at least on average and weighted by marginal utility.

Indeed, we know the value of debt is finite. So, our job must be to *interpret* the observed finite value of debt in a sensible present value formula, not to decide if the value of debt should be infinite.

4.1 Liquidity

Start with liquidity. A liquidity value of government debt can drive down its rate of return, to produce $r < g$.

The simplest example is a government that finances itself entirely by non-interest-bearing money. This government can run slight deficits forever, printing money to satisfy economic growth and inflation. Here, $r = -\pi$, and clearly $-\pi < g$ so this is a case of $r < g$. But it is obviously a limited opportunity. A big fiscal expansion from printing money quickly hits the revenue-maximizing inflation rate. Any significant deficit must still be repaid by surpluses.

We start with

$$\frac{dM_t}{dt} = -P_t s_t, \tag{2}$$

primary deficits are financed by printing money. There is a steady state with constant $M/(Py)$ at

$$\frac{M}{Py}(\pi + g) = -\frac{s}{y} \tag{3}$$

Massage (2) a bit, and we can integrate forward, discounting by the risk free rate, which is the marginal rate of substitution in this perfect-foresight constant-growth economy, to write

$$\frac{M_t}{P_t y_t} = E_t \int_{\tau=t}^{T} e^{-\left(r^f - g\right)(\tau - t)} \left(\frac{s_\tau}{y_\tau} + i_\tau \frac{M_\tau}{P_\tau y_\tau} \right) d\tau + E_t e^{-\left(r^f - g\right)(T-t)} \frac{M_T}{P_T y_T}. \tag{4}$$

I assume $r^f > g$. The point is to generate a lower return on government debt $r = -\pi < g$ and to show how the two approaches differ. Both terms converge as we take $T \to \infty$.

The real value of government debt equals the present value of surpluses, including the interest savings generated by the liquidity benefit of money, treated as a flow. This seigniorage revenue can finance a steady primary deficit $s < 0$ as given by (3). The combined surplus term in (4) remains positive,

$$\frac{s}{y} + i\frac{M}{Py} = (r^f - g)\frac{M}{Py}.$$

But, being a present value, (4) makes clear that a substantial rise in deficits must be repaid by later surpluses. If the government does not wish a large inflation, those deficits would typically be financed by adding interest-bearing debt,

$$\frac{b_t}{y_t} + \frac{M_t + B_t}{P_t y_t} = E_t \int_{\tau=t}^{\infty} e^{-\left(r^f - g\right)(\tau - t)} \left(\frac{s_\tau}{y_\tau} + i_\tau \frac{M_\tau}{P_\tau y_\tau} \right) d\tau$$

Here I add both real b and nominal B debt. Again, the transversality condition means that the limiting term goes to zero. Large deficits would be paid for by issuing such interest bearing debt, which pays $r^f > g$. We have an example in which the marginal $r = r^f > g$, though the average $r = -\pi < g$.

We can also discount by the return on government debt $r = -\pi$. Now we get

$$\frac{M_t}{P_t y_t} = E_t \int_{\tau=t}^{T} e^{(\pi+g)(\tau-t)} \frac{s_\tau}{y_\tau} d\tau + e^{(\pi+g)(T-t)} \frac{M_T}{P_T y_T}. \tag{5}$$

Now the terminal condition explodes. Since the left hand side is finite, the present value condition also explodes negatively.

Now both (4) and (5) are correct. From (2) you get to either

$$\frac{d}{dt}\left(\frac{M_t}{P_t y_t}\right) + \frac{M_t}{P_t y_t}(g - r_t^f) = -\frac{s_t}{y_t} - i_t \frac{M_t}{P_t y_t}$$

or

$$\frac{d}{dt}\left(\frac{M_t}{P_t y_t}\right) + \frac{M_t}{P_t y_t}(\pi + g) = -\frac{s_t}{y_t}$$

With $i = r^f + \pi$, these are the same. Integrate one or the other forward. The question is, which is more useful or insightful? Is it more useful to think of the liquidity services of money as providing a convenience yield flow, seignorage in the form of a lower interest cost of debt, which we discount at the real interest rate? Or is it more insightful to think of the liquidity services of money as lowering the discount rate, and then say that government debt is a "bubble" that can be "mined" for deficits?

I prefer the former. The latter can lead you mistakenly think the mine is infinite. The two elements explode in exactly offsetting directions. Though the integral explodes, surpluses themselves do not explode. You can miss the fact that substantial surpluses still need to be repaid.

The terminal condition converges in (4) but not necessarily in (5), because *The* *transversality condition holds discounting with the marginal rate of substitution,*

$$E_t\left[e^{-\rho(T-t)}\frac{u'(c_T)}{u'(c_t)}\frac{M_T}{P_T}\right] = E_t\left[e^{-r^f(T-t)}\frac{M_T}{P_T}\right] = 0$$

The "transversality condition" does not necessarily hold discounting with the ex-post return. The right hand sides of (5) may or may not converge, depending on parameter values.

A mathematician would also say that in the latter case we are simply solving the integral the wrong way. We should solve backward to express debt as an accumulation of past deficits, cumulated at the rate of return.

$$\frac{M_t}{P_t y_t} = E_t \int_{\tau=-T}^{t} e^{-(\pi+g)(t-\tau)}\frac{s_\tau}{y_\tau}d\tau + e^{-(\pi+g)(t-T)}\frac{M_T}{P_T y_T}. \tag{6}$$

This is also correct, but not very insightful.

4.2 Discount rates vs. rates of return

Here is the fundamental technical problem: *The transversality condition does not hold with all one-period discount factors.* One can always discount one-period payoffs with the ex-post rate of return, as with marginal utility or the stochastic discount factor. Moving to discrete time for clarity, while the stochastic discount factor is

$$1 = E_t\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}R_{t+1}\right),$$

we can use the ex-post return as an alternative discount factor,

$$1 = E_t\left(R_{t+1}^{-1}R_{t+1}\right).$$

It does not follow that one can always discount infinite streams of payoffs with the ex-post return or other alternative discount factors. It can happen that the present value of cashflows, discounted by the stochastic discount factor, is finite and well-behaved, i.e. that

$$p_t = E_t\sum_{j=1}^{T}\frac{\beta^j u'(c_{t+j})}{u'(c_t)}d_{t+j} + E_t\frac{\beta^j u'(c_{t+j})}{u'(c_t)}p_{t+T} \to E_t\sum_{j=1}^{\infty}\frac{\beta^j u'(c_{t+j})}{u'(c_t)}d_{t+j},$$

yet if we attempt to discount using returns,

$$p_t = E_t \sum_{j=1}^{T} \prod_{k=1}^{j} \frac{1}{R_{t+k}} d_{t+j} + E_t \prod_{k=1}^{T} \frac{1}{R_{t+k}} p_{t+T},$$

the two terms explode in opposite directions. It doesn't *always* happen. For example, the standard consumption claim in an i.i.d. economy generates convergent terms in both representations. But it can happen, depending on parameters. It's very useful to discount with ex-post returns, but convergence is a second, parameter-dependent issue. In the second equation you can see a stochastic version of $r < g$ as directly the condition that the terminal value converges, and hence the condition that you can discount using ex-post returns.

Uncertainty is key to this possibility in a world without frictions. In a world of certainty, the stochastic discount factor is the same as the risk free rate is the same as the ex-post return. To understand the bubble, then you must understand that it doesn't *always* explode. The combinations of high terminal value and low cumulative return that generate a bubble are states of nature with low marginal utility.

4.3 A lognormal example

We can see the error of using perfect foresight logic in a very simple model. Write the equation that debt equals the present value of surpluses as

$$\frac{B_t}{Y_t} = E_t \left(\sum_{j=1}^{\infty} \beta^j \frac{u'(C_{t+j})}{u'(C_t)} \frac{Y_{t+j}}{Y_t} \frac{S_{t+j}}{Y_{t+j}} \right)$$

It is tempting but incorrect to move the expectation sign inside the sum. If we do that, we obtain

$$\sum_{j=1}^{\infty} E_t \left(\beta^j \frac{u'(C_{t+j})}{u'(C_t)} \right) E_t \left(\frac{Y_{t+j}}{Y_t} \right) E_t \left(\frac{S_{t+j}}{Y_{t+j}} \right) = \sum_{j=1}^{\infty} \frac{1}{(r-g)^j} E_t \left(\frac{S_{t+j}}{Y_{t+j}} \right)$$

where the right hand equality assumes that consumption and GDP growth are independent over time. But this is incorrect – it leaves out the covariance terms.

To see that the true present value can converge while the mistaken one explodes, consider power utility and lognormal consumption, let $C = Y$ and assume a constant S/Y. Now the terms of the correct formula are

$$E_t\left[\beta^j\left(\frac{C_{t+j}}{C_t}\right)^{1-\gamma}\right]\frac{S}{Y} = e^{(1-\gamma)g+(\gamma-1)^s\sigma^2/2}\frac{S}{Y}$$

While the $r-g$ version is

$$E_t\left[\beta^j\left(\frac{C_{t+j}}{C_t}\right)^{-\gamma}\right]E_t\left[\left(\frac{C_{t+j}}{C_t}\right)\right]\frac{S}{Y} = e^{(1-\gamma)g+\gamma^2\sigma^2/2}\frac{S}{Y}.$$

With $\gamma > 1$ we have $(\gamma-1)^2 < \gamma^2$. Thus, it is entirely possible that

$$e^{(1-\gamma)g+(\gamma-1)^s\sigma^2/2} < 1 < e^{(1-\gamma)g+\gamma^2\sigma^2/2}$$

In this circumstance, the perfect foresight version – using the observed risk free rate r and average output growth rate g to discount – will indicate an explosive present value, where in fact the correctly discounted present value is finite.

5 Bohn's example

To see how uncertainty in a frictionless world can generate the $r < g$ possibility and present-value formula trouble for government debt, I adapt an example from Bohn (1995).

Suppose consumption growth is i.i.d., and there is a representative consumer with power utility. The value of the consumption stream is

$$p_t = c_t E_t \sum_{j=1}^{\infty}\beta^j\left(\frac{c_{t+j}}{c_t}\right)^{1-\gamma}$$

$$\frac{p_t}{c_t} = \sum_{j=1}^{\infty}\beta^j\left[E\left(\Delta c^{1-\gamma}\right)\right]^j = \frac{\beta\left[E\left(\Delta c^{1-\gamma}\right)\right]}{1-\beta\left[E\left(\Delta c^{1-\gamma}\right)\right]} \tag{7}$$

where $\Delta c_{t+1} \equiv c_{t+1}/c_t$. Assume that $\beta\left[E\left(\Delta c^{1-\gamma}\right)\right] < 1$, with the result that expected utility is finite. The risk free rate is

$$\frac{1}{1+r^f} = E\left(\beta\Delta c_{t+1}^{-\gamma}\right).$$

We also need to assume that consumption growth is volatile enough to drive the risk free rate down below the growth rate,

$$1 + g = E(\Delta c_{t+1}).$$

Such consumption volatility is not realistic, but this is an example.

Now, suppose the government keeps a constant debt/GDP ratio. At each date t it borrows an amount equal to GDP, c_t, and then repays it the next day, paying $(1+r^f)c_t$ at time $t+1$. (To be precise here, you should check that time-t contingent claim value of the promise to pay $(1+r^f)c_t$ indeed c_t, i.e. $E_t\left(\beta\Delta c_{t+1}^{-\gamma}(1+r^f)c_t\right) = c_t$.) The primary surplus is then

$$s_t = (1+r^f)c_{t-1} - c_t.$$

Now, the end-of-period value of government debt at time t, just after the government has borrowed c_t is obviously, $b_t = c_t$. Our job is to express that fact in terms of sensible present value relations.

If we construct a present value of surpluses, discounting properly with marginal utility, we obtain

$$b_t = E_t \sum_{j=1}^{T} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} s_{t+j} + E_t \beta^T \left(\frac{c_{t+T}}{c_t}\right)^{-\gamma} c_{t+T}$$

$$= E_t \sum_{j=1}^{T} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} \left[(1+r^f)c_{t+j-1} - c_{t+j}\right] + E_t \beta^T \left(\frac{c_{t+T}}{c_t}\right)^{-\gamma} c_{t+T}$$

It takes just a little work to boil all this back down to

$$b_t = \left[c_t - E_t \beta^T \left(\frac{c_{t+T}}{c_t}\right)^{-\gamma} c_{t+T}\right] + E_t \beta^T \left(\frac{c_{t+T}}{c_t}\right)^{-\gamma} c_{t+T} = c_t. \quad (8)$$

The present value of borrowing c_{t+j} and repaying $(1+r^f)c_{t+j}$ the next period is zero, so only the first term $(1+r^f)c_t$ at time $t+1$ survives. The last term converges to zero, via the transversality condition. (If you want to be picky, you can take a few more steps and start with b_{t+T} on the right hand side.)

However, the value of this claim *cannot* be represented by the expected value of its cashflows discounted at its ex-post return when $r^f < g$. The one-period government debt portfolio return is r^f. The return on the government debt claim is also the risk free rate $(1 + r^f)$. Attempting such a present value,

$$b_t = \sum_{j=1}^{T} \left(\prod_{k=1}^{j} \frac{1}{R_{t+k}} \right) s_{t+j} + \left(\prod_{k=1}^{T} \frac{1}{R_{t+k}} \right) b_{t+T} =$$

$$= \sum_{j=1}^{T} \frac{(1 + r^f)c_{t+j-1} - c_{t+j}}{(1 + r^f)^j} + \frac{1}{(1 + r^f)^T} c_{t+T}$$

$$b_t = \left(c_t - \frac{c_{t+T}}{(1 + r^f)^T} \right) + \frac{c_{t+T}}{(1 + r^f)^T}.$$

Taking expected value,

$$b_t = c_t \left(1 - \frac{(1 + g)^T}{(1 + r^f)^T} \right) + c_t \frac{(1 + g)^T}{(1 + r^f)^T}. \tag{9}$$

If $r^f < g$ the present value of cashflows term builds to negative infinity, and the terminal value builds to positive infinity.

Now compare the present value discounted using marginal utility, (8) to the present value discounted using the ex-post return (9). Both equations are correct. Which is more useful? At a minimum, the latter invites mistakes. Seeing an exploding terminal condition, one is tempted to find bubbles of infinite value to mine. But don't forget that the present value condition explodes in the other direction. The present value explodes though all the elements in it are finite. And this government never does anything fancy, it just keeps a steady 100% debt to GDP ratio.

6 Ex post rather than present values

OK, you say, discount using marginal utility and present value formulas converge. But the government still can borrow at r^f and roll over debt forever, no? We sort of know the answer is no once we have a present value. But it's important to spell out what goes wrong. Flow analysis at least complements present values, and often makes it more salient.

The answer is no, because *growth is stochastic.* So though $r^f < g = E(\Delta c)$ means that the government will grow out of debt *on average*, but there now states of nature in which growth will persistently disappoint. Then the government will have to raise surpluses, and do so at the most painful time, because consumption is low and marginal utility is high.

Suppose in this i.i.d. consumption growth economy, the government borrows 100% of GDP once, and then tries to simply roll over the debt at $r^f < g$. Figures 4 and 5 plot what happens. (I use parameter values $g = E(\log \Delta c) = 3\%$, $\gamma = 2$, $\delta = 0$, $\sigma = 0.15$, which generate $r^f = exp(\delta + \gamma g - 1/2\gamma^2\sigma^2) = 1.5\%$ I plot draws at the 1, 5, 50, 95 and 99 percentiles of terminal consumption.)

Since $r^f < g$, you see in the solid lines of Figures 4 and 5 that in a perfect certainty calculation growth outstrips the accumulating debt, and the debt to GDP ratio smoothly declines. But that doesn't always happen! The plots show two draws in which consumption growth disappoints, debt outstrips consumption, and the debt to GDP ratio rises spectacularly. Choose your favorite maximum debt to GDP ratio – 300, or 800 – and in these draws we discover the need to repay a massive debt with taxes, and just at the worst time because we have suffered an economic disaster, having missed what should have been 300% cumulative growth.

So, the one-time fiscal expansion, with "no fiscal cost," evaluated with perfect certainty formulas, is a bet. It is the classic strategy of writing an out-of-the-money put option, that fails in bad times, and calling it arbitrage. It may well work, if you get lucky. It may also fail at the worst moment.

Though on average g beats r^f, it does not do so weighted by marginal utility, which is why the transversality condition fails in this example of borrowing with no repayment.

$$E_0\left[\beta^T \frac{u'(c_T)}{u'(c_0)} b_T\right] = E_0\left[\beta^T \frac{u'(c_T)}{u'(c_0)} b_0 (1+r^f)^t\right] = E_0\left[\beta^T \frac{u'(c_T)}{u'(c_0)} b_0 \frac{1}{\beta^T \frac{u'(c_T)}{u'(c_0)}}\right] = b_0.$$

7 Bottom line

These examples are simple and not quantitatively realistic. Reis (2021) contains a detailed though rather complex model with firm uncertainty, production, and financial frictions generating a liquidity premium for government bonds. The complexity is to some extent necessary to micro-found liquidity. It also helps to generate realistic parameter

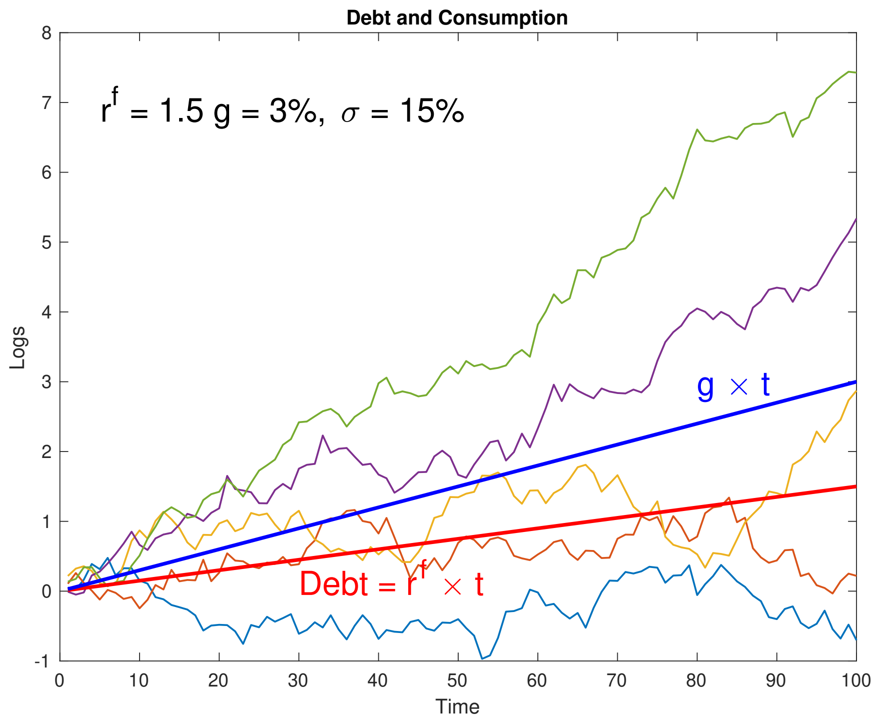

Figure 4: Path of perpetually rolled over debt, and consumption.

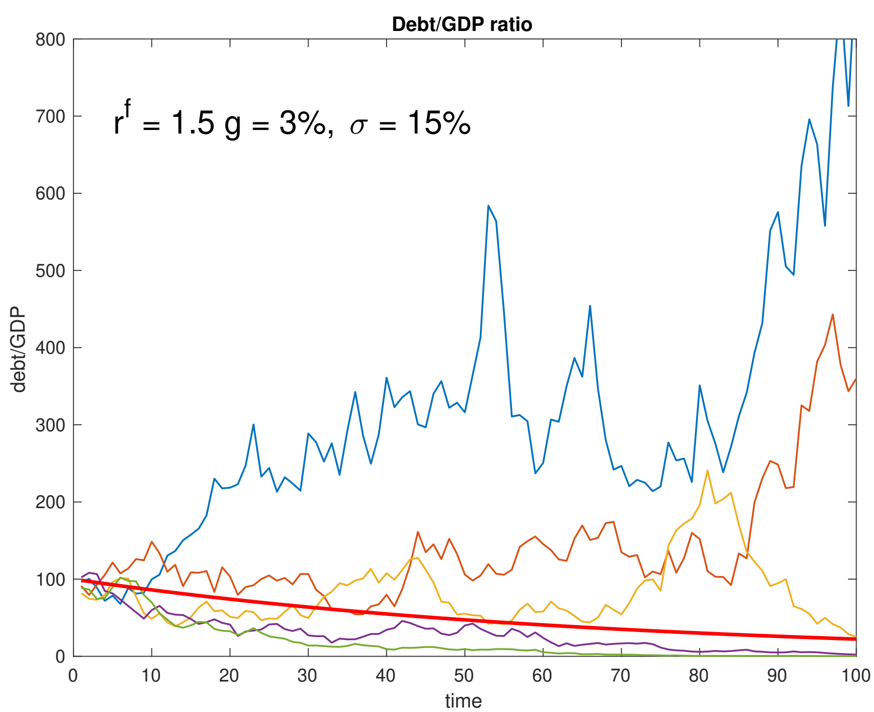

Figure 5: Paths of debt to GDP (consumption) ratio.

configurations for which r is low. To overcome

$$r = \delta + \gamma(g - n) - 1/2(\gamma)(\gamma - 1)\sigma^2$$

I had to assume an unrealistically large σ. One needs either different preferences or a more complex model to generate $r < g$ from uncertainty realistically. Blanchard (2019) includes a detailed review.

But the basic point is much more general, and as usual microfounded detail and quantitative realism hide how important that basic point is.

The bottom line:

$r < g$ is like seignorage, allowing a small steady deficit. But $r < g$ is irrelevant for the big issues of US fiscal policy.

Despite $r < g$, *large deficits still need to be repaid with primary surpluses*, at least in marginal utility weighted terms. The grow-out-of-debt strategy is like writing out of the money put options and calling it arbitrage.

Present value formulas can express the finite value of debt, and the fact that large deficits must be repaid by lager surpluses. One needs to take present value formulas carefully however. When discounting by marginal utility works, discounting by ex-post rates of return can generate formulas with offsetting infinities.

If our fiscal situation is so dire, why do bond investors still lend money to the government at astonishingly low rates? First, recall that bond markets have never seen trouble coming, or waning. They did not see inflation in the 1970s, they did not see disinflation in the 1980s, they did not see Lehman or Greece coming. If you knew a crisis was going to happen tomorrow, it would happen today. Second, and more importantly, the CBO forecasts are not conditional means. They are "here is what will happen if you don't do something about it" calculations. Surely the USA will not borrow and spend itself into a debt crisis. Fixing our structural budget problems is not hard as a matter of economics, or, really, once everyone recognizes it has to be done, as a matter of politics. I hazard bond markets assume the US will, as usual, get around to doing the right thing after we try everything else. Let us hope so.

References

Blanchard, Olivier. 2019. "Public debt and low interest rates." *American Economic Review* 109:1197–1229.

Bohn, Henning. 1995. "The Sustainability of Budget Deficits in a Stochastic Economy." *Journal of Money, Credit and Banking* 27 (1):257–271. URL http://www.jstor.org/stable/2077862.

Friedman, Milton. 1968. "The Role of Monetary Policy." *The American Economic Review* 58:1–17.

Reis, Ricardo. 2021. "The Constraint on Public Debt when $r < g$ but $g < m$." *Manuscript* .