11 Week 5 Asset pricing theory – detailed notes

Note: The reading from *Asset Pricing* lays out asset pricing theory in a careful way. These notes are not a substitute for the reading. These notes simply cover the way I plan to discuss the material in class.

11.1 Motivation

1. Week 1: Expected excess returns = risk premia vary *over time*. Why? Something about time-varying willingness to bear risks over the business cycle. Week 2: Expected excess returns = risk premia vary *across assets* ("cross-section") at any point in time. Why? Something about "state variables of concern to investors."

2. Note: The question we are asking is not "Where do I find great unexploited opportunities?" The question we are asking is, "Why might the average investor be scared of "value risk," or more scared at some times than others, and thus does not take these opportunities?" "In what sense might they really not be "opportunities" but "rewards exactly balanced by risks? Why are these patterns in *equilibrium* expected returns? At least ask this question before trading! If you can figure out why everyone else is scared of the apparent opportunity, that puts you in a better position to know if the risk/reward is right for you.

3. "Explanations" we have seen

   (a) CAPM. Higher average returns "because" higher betas

   $$\begin{aligned} E(R^{ei}) &= \beta_i \lambda_m \\ E(R^{ei}) &= \beta_i E(R^{em}) \end{aligned}$$

   (b) FF: this fails. They "explain" with FF3F betas

   $$\begin{aligned} E(R^{ei}) &= b_i \lambda_m + h_i \lambda_{hml} + s_i \lambda_{smb} \\ E(R^{ei}) &= b_i E(rmrf) + h_i E(hml) + s_i E(smb) \end{aligned}$$

   (c) Why is this not an "explanation?"

   $$E(R^{ei}) = (stock_i)\lambda_m + \log(size_i)\lambda_s + \log(beme_i)\lambda_{hml}$$

   A: This is a good *description* but not a *model,* or *explanation. Models, explanations* seem to need *regression coefficients* on the right hand side to "explain" the pattern of average returns. But why really?

   (d) A hint: regression coefficients have something to do with markets in equilibrium, with keeping you from making a fortune. If $E(R^{ei}) = 16\%$ while the market yields $E(R^{em}) = 8\%$, the observation that $\beta_i = 2$ means you can't buy the asset, short the market, and earn the return difference.

4. →Where do CAPM, FF3F come from? What constitutes an "explanation?" What is the question to which this is the answer? Persuade yourself that "beta" really does mean "risk" – and understand when (and for whom) it does not.

5. $\rightarrow$ Theory to understand risk premia.
6. $\rightarrow$ Understand Fama and French "APT" vs. "ICAPM," "state variables for xyz" discussion
7. Bottom line: Assets pay more on average if they tend to do badly when people are "hungry", or more desperate for money.
8. Method: Just a little algebra goes a long way!

11.2 APT

- Objective: Understand why *factor structure* – high R^2 in time-series regressions – implies *factor pricing* – small alphas in the cross-sectional relation between average returns and betas.

1. Suppose you have N test assets R^{ei}, excess returns (for example the FF 25) and K factors f^i, also excess returns (for example rmrf, hml, smb). Run a time-series regression

$$R^{ei}_{t+1} = \alpha_i + \beta_{i1} f^1_{t+1} + \beta_{i2} f^2_{t+1} + \varepsilon^i_{t+1}.$$

We want to conclude

$$E(R^{ei}) = \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 = \beta_{i1}E(f^1) + \beta_{i2}E(f^2)$$

i.e. $\alpha_i = 0$.What does it take to reach this conclusion?

2. "Exact APT:" If $\varepsilon = 0$, we must see $\alpha = 0$ or there is arbitrage. Why? Go long R^{ei}_{t+1}, go short $\beta_{i1} f^1_{t+1} + \beta_{i2} f^2_{t+1}$. The portfolio return is

$$R^{ep}_{t+1} = R^{ei}_{t+1} - \left(\beta_{i1} f^1_{t+1} + \beta_{i2} f^2_{t+1}\right) = \alpha^i$$

This is not random! It has zero cost, and a positive return!

   (a) Example: A fund with zero tracking error, return 10 bp above S&P500 index? Short index, long fund, earn 10bp for free.

   (b) It's an arbitrage, let's buy!....No, if we're describing a *market at equilibrium*, then arbitrage is gone. People trying to do this push up the price of the fund and down the price of S&P500, until $\alpha = 0$. In equilibrium, so long as investors are smart enough to get there first and snap up arbitrages, we see $\alpha = 0$.

   (c) If the R^2 were 100%, The FF regressions would be describing how the returns on the 25 portfolios could be exactly replicated by returns of the assets. So the means on those portfolios should be the same as the means of the hedge portfolio.

3. "Approximate APT." If ε is "*small*," α should be "*small.*"

   (a) Large α does not imply arbitrage, but maybe a very good deal? Let's look at the portfolio long the stock and short its factor content. This gives the alpha but now with some error too,

$$R^{ep}_{t+1} = R^{ei}_{t+1} - \left(\beta_{i1} f^1_{t+1} + \beta_{i2} f^2_{t+1}\right) = \alpha^i + \varepsilon^i_{t+1}$$

$$E(R^{ep}_{t+1}) = \alpha^i; \quad \sigma(R^{ep}_{t+1}) = \sigma(\varepsilon^i)$$

$$\text{Sharpe} = \frac{E(R^{ep}_{t+1})}{\sigma(R^{ep}_{t+1})} = \frac{\alpha^i}{\sigma(\varepsilon^i)}$$

(Coming: What's the optimal investment in the hedge portfolio?

$$\text{weight} = \frac{\alpha_i}{\gamma\sigma^2(\varepsilon^i)}$$

but you don't know that yet.) Conclusion: *If $\alpha >> \sigma(\varepsilon)$ there are really good deals to be had.*

(b) Ruling those out (equilibrium, remember?)

$$\alpha^i < (\text{max surviving Sharpe ratio}) \times \sigma(\varepsilon^i)$$

we conclude that *alphas should be small when residual variance is small*

(c) When is $\sigma(\varepsilon)$ small?

$$\begin{aligned} R^e_{t+1} &= \alpha + \beta f_{t+1} + \varepsilon_{t+1} \\ R^2 &= 1 - \frac{\sigma^2(\varepsilon)}{\sigma^2(R^e)} \end{aligned}$$

Alphas should be small when the R^2 in the time series regression is large. Let's look at FF table 1...Hey, this is an APT!

4. Comments on the APT

(a) The APT bottom line: "where there is mean, there must be covariance" (or, there would be astronomical Sharpe ratios.) If the value stocks all earn high average returns, the value stocks must *all move together*, so that diversification among the value stocks does not give you the high mean without risk. The "factor" (hml) still gives mean, with a risk uncorrelated with the market return – but that's "factor risk" with a limited Sharpe ratio, not "arbitrage."

(b) The APT is typically useful for *portfolios* (FF 25) or *managers* (of diversified portfolios or with tracking error constraints that give high R^2) but not for *individual stocks.* Individual stocks have lower R^2. *The APT applies to "well diversified portfolios"* (Those with high R^2)

(c) A very important use of this construction: *The time-series regression coefficients are a portfolio.* How do you optimally hedge a position? A: Run a regression against factors. The coefficients tell you how much to short the factors to get the smallest basis risk ε!

(d) What have we done? We explained *test asset* portfolios *given* factor portfolios. Who says FF rmrf, smb, hml portfolios are priced right? That's not the question; the question is *given* those factors, are the 25 priced right? "Behavioralists" may still be right if hml, smb, and umd (!) are priced wrong. It is very useful though, a factor model reduces our quest to understanding $E(rmrf)$ $E(hml)$ $E(smb)$ $E(umd)$ not every strategy in isolation.

(e) *Absolute* vs. *Relative* pricing. Rather than a "theory of everything," we *extend* known prices to value something else. (An idea taken to its limit in Black-Scholes, where we can price options from stock and bonds.) This is a BIG idea. APT, Black-Scholes option pricing (price of option given the stock and the bond) all term structure models (price of bonds, caps, swaps, given the term structure "factors"), all "pricing by arbitrage," using "comparables" for corporate valuation.

(f) Surely "relative pricing" is the right approach for most practical applications. You don't care why the S&P500 does what it does, you want to know "can this manager (strategy, etc.) beat the S&P500?" Analogy: What's the value of a burger at McDonalds? 1) Cost to raise a cow, ... 2) What's a burger at Wendy's? (and then adjust). This is less useful for the grand "explain" project, but much more useful in practice.

5. Motivation for more theory

   (a) What about assets that do not have high R^2 – not easily replicated by a few standard indices?

   (b) Why is $E(rmrf)$, $E(hml), E(smb), E(umd)$, so high? Even the CAPM says "your expected return is high because you have a large beta and the expected market return is 7%." But why is the market expected return 7%? Isn't this supremely huge? We can't use the CAPM to answer that question! FF say the 25 size/bm portfolios, and a range of additional anomalies, are "explained" by their exposure to the *hml* and *smb* factors. But why are $E(hml)$ and $E(smb)$ so high? The FF model can't explain that.

   (c) Needed: an *economic theory of risk premiums.*

11.3 Utility and Asset Pricing

Rather than APT – price one thing in terms of another – we build the generic theory of asset pricing. How do you price any asset?

Basic question: What do investors want? What is the value of a security to an investor? Describe the investor by a utility function and find out.

*=Optional material.

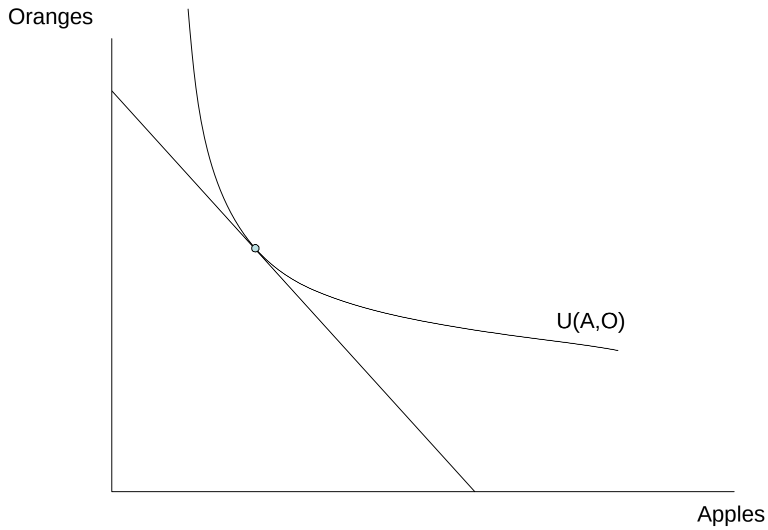
1. You have a *cash flow* – dividends (stocks), coupons (bonds), rent (real estate), profits (build a factory), call option payoff. .

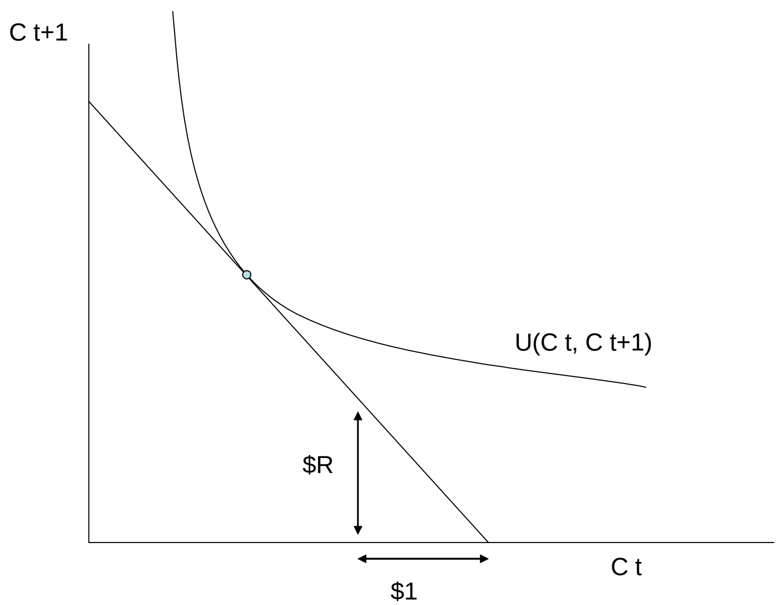
   (a) What is its *value?* What are the effects *risk* and *time.*

   (b) How does value change if the world changes – $d\text{Value}/dz$? If you know this, you know how to do risk management.

   (c) Approach: apply apples and oranges microeconomics to finance.

   (d) *Preview:



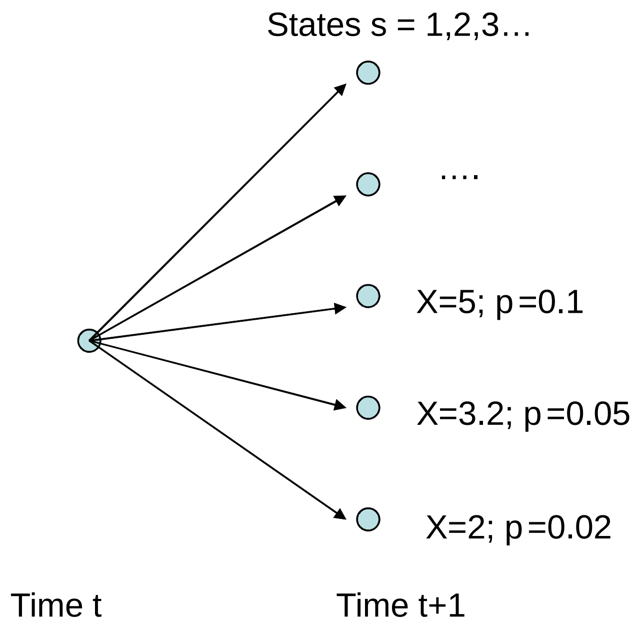


2. Payoffs

   (a) Payoff x_{t+1} tomorrow. (For stocks, $x_{t+1} = p_{t+1} + d_{t+1}$)

   (b) x_{t+1} is a *random variable*, like a coin flip – we don't know at t what it will be, though we can assign probabilities to the possible outcomes.

   (c) Randomness: you can think of x_{t+1} (and anything else that happens at $t+1$) as taking on different values in different *states of the world.* For example, $E(x_{t+1}) = \sum_s \pi(s)x(s)$



(d) Our question: Price or value p_t today of this payoff?

(e) (A common confusion: *Payoff* $x_{t+1}$is not *profit* $x_{t+1}-p_t$. Remember call payoff diagrams?)

11.3.1 Utility functions

- Objective: Understand the utility function $U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})]$. Meet the common utility function with $u'(c) = c^{-\gamma}$.

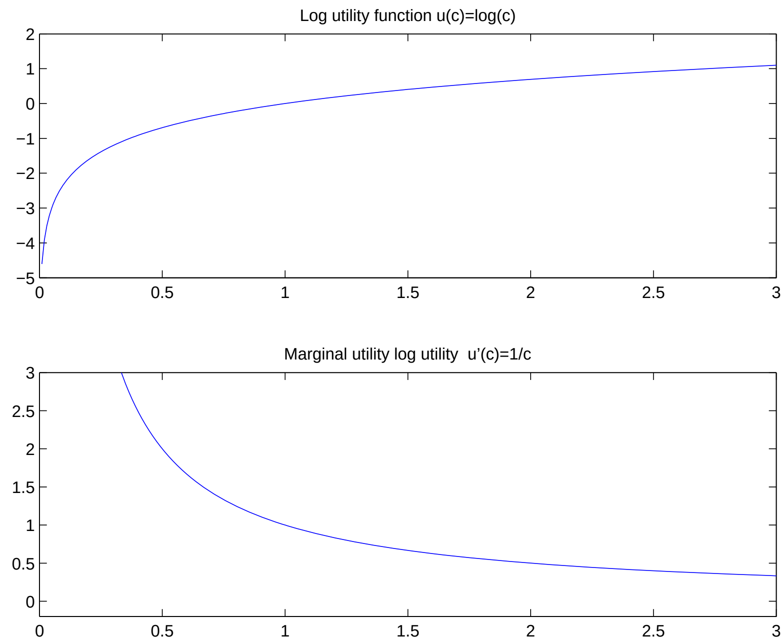
1. Value to who? An investor. We describe what the investor wants by a *utility function.*

$$U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})]$$

The point of a utility function is to model (capture) investor's aversion to *risk* and *delay,* and appropriately discount prices. There is no absolute theory of asset pricing, what assets "should" be worth. Asset pricing is all about quantitative psychology, what are people willing to pay for assets? That depends on how impatient and risk averse they are. The utility function just gives us a good way of seeing how impatience and risk aversion impact asset prices.

2. Example utility function: Log

$$u(c) = \ln(c);\; u'(c) = \frac{1}{c}$$



3. Utility functions

   (a) $u(c) =$ 'happiness' $u'(c) =$ 'hunger'

   (b) The *Level* of $u(c)$ doesn't matter for anything. It's ok if $u(c)$ is negative. (-20° is warm in Alaska) Maximizing $u(c)$ gives the same c as maximizing $\{u(c) + 10\}$.

   (c) $u(c)$ rises, $u'(c) > 0$. People always want more.

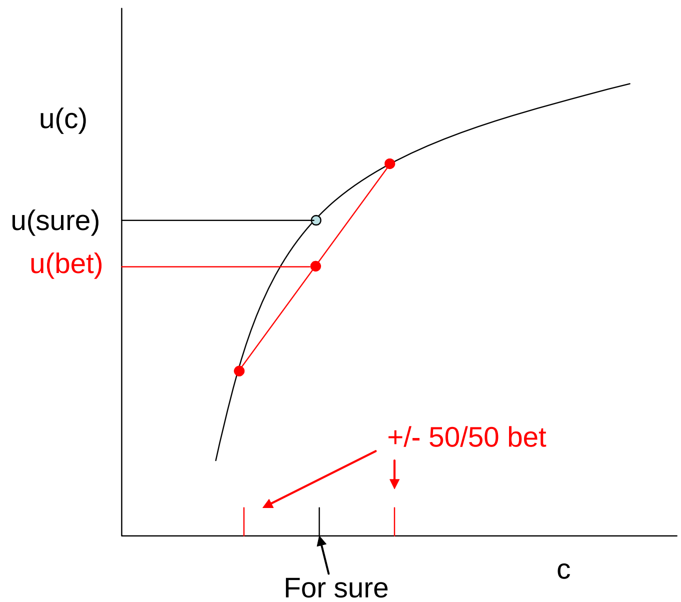
   (d) $u'(c)$ declines with c, $u(c)$ concave. Hunger declines as you eat more.

   (e) β is a number, typically 0.95 or so. People prefer money now to later, they dislike *delay*.. (β has nothing to do with CAPM beta – two uses of the same letter.) β captures their impatience.

   (f) c_{t+1} is random; you don't know at time t how things will turn out, what c_{t+1} will be.

   (g) Thus utility of *random* consumption is $E_t u(c_{t+1})$. Example: 50/50 bet. $E\left[u(c)\right] = \frac{1}{2}u(\bar{c} + x) + \frac{1}{2}u(\bar{c} - x)$. The utility function describes how you feel *ahead of time* about the random outcome you will receive in the future

   (h) Concavity and expected utility → people dislike *risk*. They would prefer to give up some consumption for sure in order to avoid a 50/50 bet.



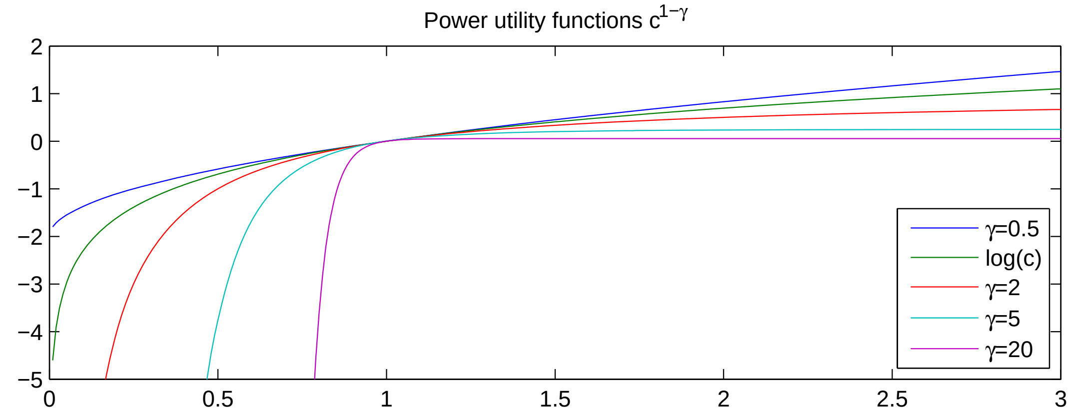
In math

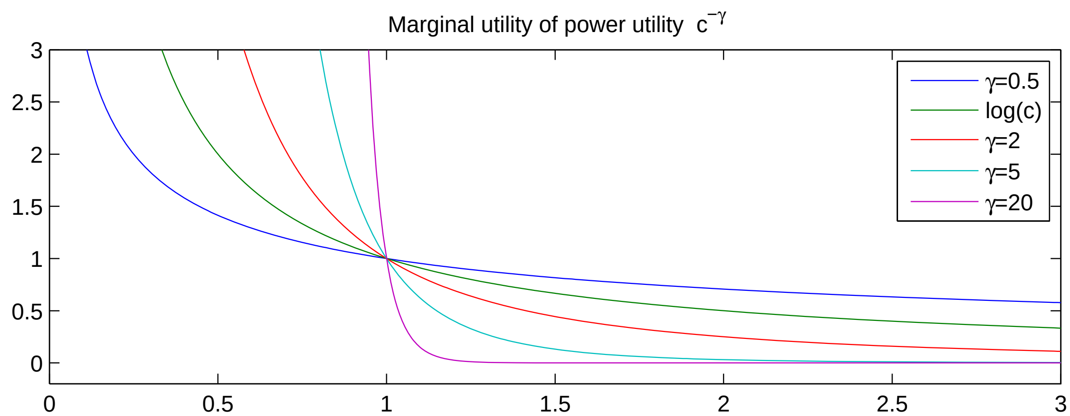
$$E\left[u(c)\right] = \frac{1}{2}u(\bar{c}+x) + \frac{1}{2}u(\bar{c}-x) < u\left[E(c)\right] = u\left(c + \frac{1}{2}x - \frac{1}{2}x\right)$$

(i) *The concave $u(c)$ (c vs. $u(c)$ graph) induces curved indifference curves over $U(c_t, c_{t+1})$ (c_t. vs. c_{t+1} graph) that look like the apple-orange case. (See above c_t vs. c_{t+1} graph) People are less and less willing to give up some c_t to get more c_{t+1}. (We'll explore this on the problem set) If $u(c) = kc$, linear, then the c_t vs. c_{t+1} indifference curves are linear too. In this case people are very willing to substitute consumption over time and take risk. As both $u(c)$ and indifference curves become more curved, people are less and less willing to take risks / move consumption over time when prices scream at them to do so. (E.g. Fall 2008)

4. A more useful functional form generalizes log; it lets you have more or less curved function = more or less risk aversion

$$\begin{aligned} u(c) &= \frac{c^{1-\gamma}}{1-\gamma} \rightarrow u'(c) = c^{-\gamma} \\ \gamma &= 1 : u(c) = \ln(c) \rightarrow u'(c) = \frac{1}{c} \end{aligned}$$





*The *coefficient of relative risk aversion*

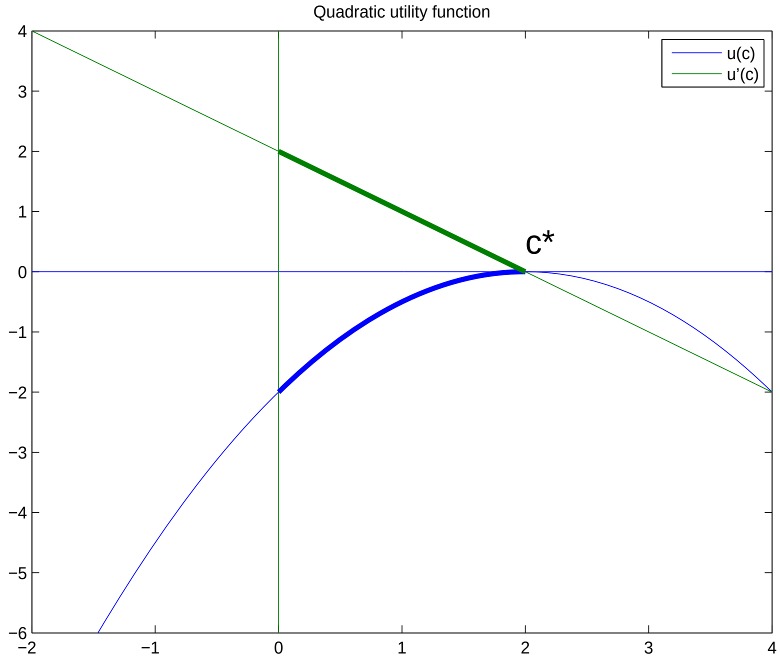
$$\gamma = -\frac{cu''(c)}{u'(c)}$$

measures how curved the utility function is, and thus how resistant people are to taking risks and to substituting consumption over time. The power utility function is also known as *constant relative risk aversion*

$$\begin{aligned} u'(c) &= c^{-\gamma} \\ u''(c) &= -\gamma c^{-\gamma-1} \\ -\frac{cu''(c)}{u'(c)} &= -\frac{-c\gamma c^{-\gamma-1}}{c^{-\gamma}} = \gamma! \end{aligned}$$

5. We also sometimes use quadratic utility Suppose utility is quadratic.

$$u(c) = -\frac{1}{2}(c^* - c)^2 \Rightarrow u'(c) = c^* - c$$



Quadratic utility is very popular, because then marginal utility is linear. It's only an approximation, though easy to work with, since it extends to negative consumption, and predicts that people want to throw things away past c^* Quadratic utility makes deriving the CAPM easy, and mean-variance portfolio theory.

11.3.2 The basic asset pricing formula

- Objective: Understand the basic asset pricing formula

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]$$

1. Question: *What is the value of payoff x_{t+1} to an investor with a utility function $u(c_t) + \beta E_t\left[u(c_{t+1})\right]$?* This formula is an answer to that question.

2. Given the properties of the payoff x_{t+1} and the investor's consumption, you can calculate the value p_t.For example, a log investor

$$p_t = E_t\left[\beta \frac{c_t}{c_{t+1}} x_{t+1}\right]$$

   If you know c_t and have an idea of the risks the consumer faces (c_{t+1}) and the asset payoff x_{t+1}

3. This is IT. *All asset pricing and portfolio theory flows from this one equation. Yes, everything. CAPM, FF3F, option pricing, bond pricing, and portfolio theory.*

4. Why? Think about buying a little more. You're sitting at this level of utility:

$$U_{before} = u(c_t) + \beta E_t\left[u(c_{t+1})\right]$$

If you buy ξ more shares, you lose $p_t\xi$ consumption today. But you gain ξx_{t+1} more tomorrow. So,

$$U_{after} = u(c_t - p_t\xi) + \beta E_t\left[u(c_{t+1} + \xi x_{t+1})\right]$$

Now, for a small extra purchase,

$$U_{after} \approx u(c_t) - u'(c_t)p_t\xi + \beta E_t\left[u(c_{t+1}) + u'(c_{t+1})\xi x_{t+1})\right]$$

so the *increase* in utility is benefit:

$$U_{after} - U_{before} \approx -u'(c_t)p_t\xi + \beta E_t\left[u'(c_{t+1})x_{t+1}\xi\right].$$

If utility increases, you should buy some more. You keep going until the cost just balances the benefit, where marginal benefit is just equal to marginal cost, or where

$$p_t u'(c_t) = \beta E_t\left[u'(c_{t+1})x_{t+1}\right]$$

5. This is the valuation *after* the investor has bought all he wants of this and all other securities. It only holds for buying "small" (infinitesimal) amounts of the security.

   (a) *Before* buying, of course,

   $$p_t < E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}x_{t+1}\right].$$

   The security is "underpriced" *to you* even if it is "correctly priced" to everyone else. For example, suppose the right thing to do is buy the market portfolio, and you've only bought half your stocks. Well, the prices are right since everyone else has bought, but the ones you haven't bought look "underpriced" to you. As you buy you become more exposed to risks – your $\{c_t, c_{t+1}\}$ vary. You get less c_t since you're investing more and your c_{t+1} becomes more correlated with the stock market x_{t+1}. You stop buying when *consumption* has adapted to this formula.

   (b) To the individual, price is fixed and this is a recipe for how to adjust consumption. As we all try to buy, however we affect prices. If consumption is given by total output, then in aggregate, this is a recipe for how equilibrium asset prices are formed given consumption. Micro and macro causality can go in opposite directions!

   (c) This is a common misconception. Individually we can sell stocks. In aggregate, all we can do is push the prices down.

   (d) If you can't make "marginal" investments, i.e. a private equity deal, then "idiosyncratic risk matters," and you can't use marginal analysis like this.

11.3.3 Discount Factor

- Objective: understand m notation and the approximation that discount factor is a linear function of consumption growth.

$$\begin{aligned} p_t &= E_t\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}x_{t+1}\right) = E_t\left(m_{t+1}x_{t+1}\right) \\ m_{t+1} &= \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = e^{-\delta}e^{-\gamma\Delta c_{t+1}} \approx 1 - \delta - \gamma\Delta c_{t+1} \end{aligned}$$

1. It's useful to separate

$$\begin{aligned} m_{t+1} &\equiv \beta \frac{u'(c_{t+1})}{u'(c_t)} \\ p_t &= E_t\left(m_{t+1} x_{t+1}\right) \end{aligned}$$

$$p = E(mx)$$

(When I leave off subscripts, understand p_t, m_{t+1}, x_{t+1})

2. *Why? We'll see many models of m, (not just $\beta u'(c_{t+1})/u'(c_t)$) and many different expressions of $p = E(mx)$. Example: in option pricing we find m to price stock, bond, rather than from consumption data. *All of asset pricing theory and practice comes down to various tricks for finding m that are useful in specific applications. This includes stocks, bonds, options, fx, real investment valuation, etc..*

3. In our example

$$\begin{aligned} u &= \ln(c) : m_{t+1} = \beta \frac{c_t}{c_{t+1}} \\ u &= c^{1-\gamma} : m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \end{aligned}$$

4. A good approximation,

$$\begin{aligned} m_{t+1} &= \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = e^{-\delta} e^{-\gamma \Delta c_{t+1}} \approx 1 - \delta - \gamma \Delta c_{t+1}; \\ m_{t+1} &\approx 1 - \delta - \gamma \Delta c_{t+1} \end{aligned}$$

where

$$\begin{aligned} \beta &= e^{-\delta} \approx 1 - \delta \\ 0.95 &\approx 1 - 0.05 \\ \Delta c_{t+1} &= \log\left(\frac{C_{t+1}}{C_t}\right) \\ 0.10 &\approx \log\left(\frac{1.10}{1.00}\right) \end{aligned}$$

i.e. $\delta = 0.05$ for 5% discount rate; $\Delta c_t = 0.01$ for 1% consumption growth rate.

5. (**If you know continuos time, these approximations are all much easier by taking Ito's lemma derivatives of $c_t^{-\gamma}$.)

6. m, $u'(c)$ measure "hunger." m, u' are high when c is low. *Hunger u' is higher in bad times when c is low.*

11.4 Classic issues in finance

- Objective: Let's *use* this theory, which will clarify it. All the classic propositions of finance follow from $p = E(mx)$

1. Interest rates. Pay \$1, Get R^f (e.g. 1.03). Thus

$$\begin{aligned} 1 &= E(mR^f) = E(m)R^f \\ R_t^f &= 1/E_t(m_{t+1}) \end{aligned}$$

   Risk free rate and consumption:

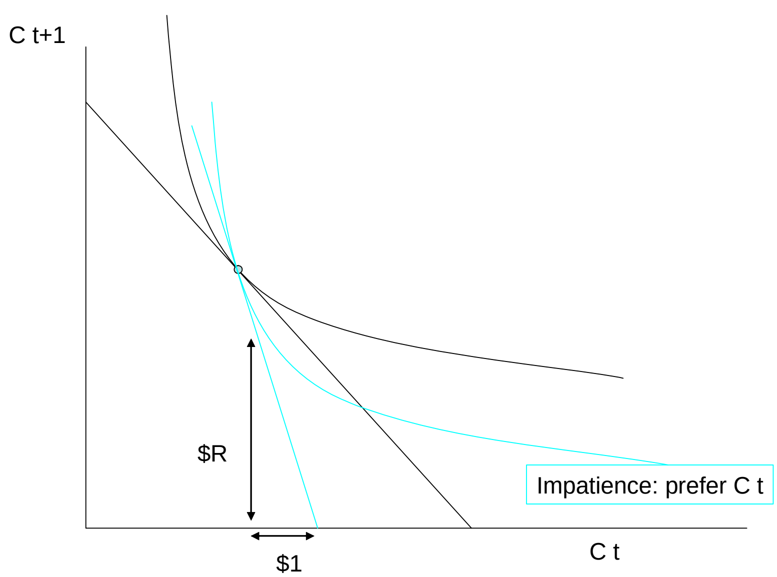
$$R_t^f = \frac{1}{\beta\left[E_t\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]} \approx \frac{1}{1-\delta-\gamma E_t(\Delta c_{t+1})} \approx 1+\delta+\gamma E_t(\Delta c_{t+1})$$

   (a) (**Notation: We call it R_t^f and R_{t+1} though both returns are from t to $t+1$. That's because we know the value of R_t^f at time t. Use notation $R_{t\to t+1}$ if you're confused. Be careful with data. The interest rate for March 31 to April 30 is often in the March 31 row, and the stock return is in the April 30 row. When you form an excess return, make sure you know what the convention is in the data you're using. The rf in ff data is dated $t+1$, i.e. in the same row with the stock returns.)

2. *When/where are interest rates high?* If people are impatient or if consumption growth is high.

   (a) Interest rates are higher if people are impatient (high δ, low β).

   (b) *Graph:



   (c) Interest rates are higher if expected consumption *growth* is high.

$$R^f \approx 1+\delta+\gamma E_t(\Delta c_{t+1})$$

      i. If people know they will be richer in the future, you must offer high rates to get them to consume less now and save.

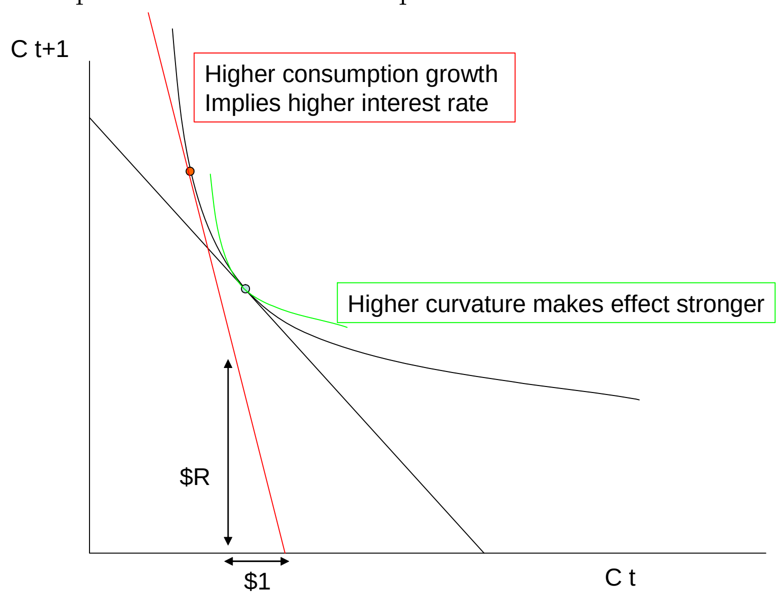
      ii. No surprise, interest rates are higher in booms (higher Δc_{t+1}) than in recessions. This is often attributed to the Federal Reserve, but maybe the Fed really has little choice in the matter. OTOH, new-Keynesian macroeconomists think the Fed sets the interest rate R^f, and consumption growth follows. Too high R^f means consumption growth is too high, so the level today is too low.

(d) *R^f is more *sensitive* to consumption growth if γ is high.

$$R^f \approx 1 + \delta + \gamma E_t(\Delta c_{t+1})$$

i. Boom: high $E_t(\Delta c_{t+1})$, high R^f. Bust: low or negative $E_t(\Delta c_{t+1})$, low R^f.

ii. How much does R^f vary? How much must you offer people to postpone consumption? $1/\gamma$ = "elasticity of intertemporal substitution." *Graph:



(e) * In this case a second-order approximation is interesting too.

$$R_t^f \approx 1 + \delta + \gamma E_t(\Delta c_{t+1}) - \frac{1}{2}\gamma^2 \sigma_t^2(\Delta c_{t+1})$$

The approximation is exact in continuous time or with lognormal consumption growth. Higher *volatility* of consumption growth makes interest rates *lower.* In more uncertain times, people want to save more. More demand for "precautionary savings" drives interest rates down.

(f) *Does Δc adjust to R^f or does R^f adjust to Δc? Both really. For you and me, R^f is given, and Δc adjusts. For a small open country, the same. For a closed economy or the world, to some extent it is R^f that adjusts.

(g) **Important point:* we are studying a market *after* it has settled down, and everyone has bought as much as they want.

(h) *This is really about *real* interest rates. What about inflation?

$$\text{real return} = R_{t+1}\frac{\Pi_t}{\Pi_{t+1}}$$

where Π_t is the CPI level (e.g. 110). Thus,

$$1 = R_t^f E\left(m\frac{\Pi_t}{\Pi_{t+1}}\right)$$

Nominal interest rates are higher if people expect a lot of inflation

3. Valuing risk.

(a) Use the definition of covariance

$$cov(m,x) \equiv E(mx) - E(m)E(x). \tag{29}$$

Thus,

$$\begin{aligned} p &= E(mx) = E(m)E(x) + cov(m,x) \\ p &= \underbrace{\tfrac{E(x)}{R^f}}_{\text{present value (time)}} + \underbrace{cov(m,x)}_{\text{risk correction}} \end{aligned}$$

(b) With the approximation

$$m_{t+1} \approx 1 - \delta - \gamma \Delta c_{t+1}$$

we get

$$p_t^i \approx \frac{E_t(x_{t+1}^i)}{R^f} - \gamma cov(x_{t+1}^i, \Delta c_{t+1})$$

(I added back i to remind you of what is specific to the asset and what's common to all assets.) The price is lower if you do well in *good* times and price is higher if you do well in *bad* times. *Prices are higher for assets that "provide insurance" against consumption risks. Prices are low for assets that, if you buy them, make your consumption more risky.* Note a "beta" beginning to appear.

(c) An example of why *co*variance is important. Suppose there are two states u, d tomorrow with probability 1/2 (As in binomial option pricing.)

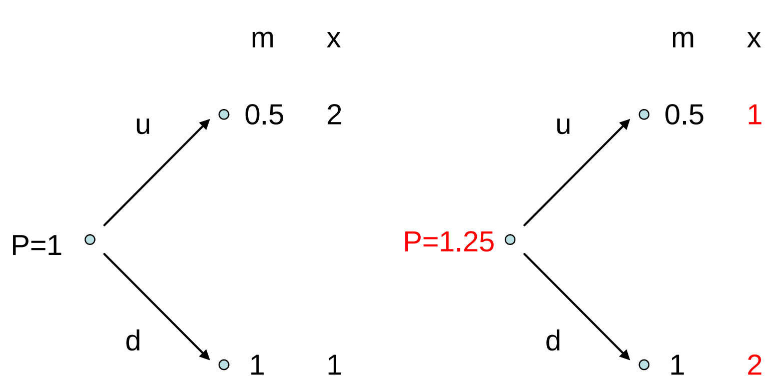
$$p_t = E(mx) = \frac{1}{2} m_u x_u + \frac{1}{2} m_d x_d.$$

u is "good times" with high c, low m. Thus, suppose $m_u = 0.5$, $m_d = 1$. (We could have $\gamma = 1$, $c_t = 1$, $c_{t+1}(u) = 2, c_{t+1}(d) = 1$, and $m_{t+1} = (c_{t+1}/c_t)^{-1}$.) Now, suppose x pays off well in "good times", If $x_u = 2$, $x_d = 1$.

$$p_t = E(mx) = \frac{1}{2} \times 0.5 \times 2 + \frac{1}{2} \times 1 \times 1 = 1.$$

But suppose we switch – same volatility but x pays off well in bad times and badly in good times. $x_u = 1, x_d = 2$.

$$p_t = E(mx) = \frac{1}{2} \times 0.5 \times 1 + \frac{1}{2} \times 1 \times 2 = 1.25.$$



Note $E(x)$, $\sigma(x)$ is the same. The payoff is worth more if the good outcome happens when m is high (hungry) rather than when m is low (full). The *same* m and the *same* x deliver *different* risk adjustments depending on $cov(m,x)$. m acts like a "price." It says that payoffs delivered in the bad state of nature (d) are worth more than payoffs delivered in the good state of nature (u).

(d) Terminology: $m =$ *stochastic discount factor*. Why? Remember the old discount factor,

$$p_t^i = \frac{E(x_{t+1}^i)}{ER^i}$$

i. $1/ER^i =$ *discount factor*. But it's different for each asset i. (Reminder: 35200 advice: use CAPM for ER^i).

ii. Our version

$$p_t^i = E(m_{t+1}x_{t+1}^i)$$

m is *stochastic* (unknown at t, inside E), and the *same for all assets.* Different covariance of m, x^i gives different risk adjustments for different assets.

4. Risk and betas.

(a) Objective: $p = E(mx)$ implies $E(R^{ei}) = -R^f cov(R^{ei}, m) = \beta_{R^{ei},m}\lambda_m$.

(b) *Excess* returns or *zero-cost portfolios* have price 0, payoff = return $R^e = R^i - R^j$, so $p = E(mx)$ reads

$$0 = E_t(m_{t+1}R_{t+1}^e.)$$

(c) To expected returns and betas. Trick: Use the definition of covariance (we want betas)

$$cov(m,x) \equiv E(mx) - E(m)E(x)$$

$$\Rightarrow E(mx) = cov(m,x) + E(m)E(x)$$

Then

$$\begin{aligned} 0 &= E(mR^e) = E(m)E(R^e) + cov(m, R^e) \\ E(m)E(R^e) &= -cov(m, R^e) \\ E(R^e) &= -R^f cov(m, R^e) \end{aligned}$$

To betas

$$\begin{aligned} E(R^e) &= \frac{cov(m, R^e)}{var(m)}\left[-R^f var(m)\right] \\ E(R^e) &= \beta_{R^e,m} \times \lambda_m \end{aligned}$$

(d) To consumption. *Important trick: (Linear factor models)* If $m_{t+1} = a - bf_{t+1}$ then

$$\begin{aligned} E(R^e) &= -R^f cov(R^e, m) = R^f \times b \times cov(R^e, f) \\ &= \frac{cov(R^e, f)}{var(f)}\left[R^f \times b \times var(f)\right] = \beta_{R^e,f}\lambda_f \end{aligned}$$

(e) *To consumption.* Using $m_{t+1} \approx 1 - \delta - \gamma \Delta c_{t+1}$, and the linear factor models trick,

$$E(R^e) \approx -cov(1 - \delta - \gamma \Delta c, R^e) \approx \gamma cov(R^e, \Delta c).$$

(I also assumed $R^f \approx 1$ which is really good for short time intervals). Using part 2 of the linear factor model trick,

$$E(R^e) \approx \beta_{R^e,\Delta c} \times \lambda_{\Delta c}$$

(f) A reminder of what this means.

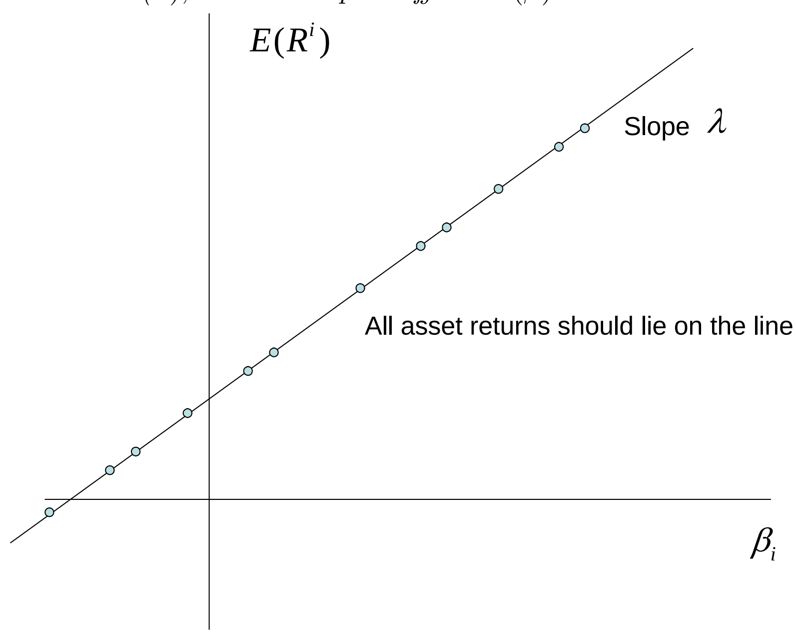
i. Run *time series regression* to find betas

$$R^i_{t+1} = a_i + \beta_{i,\Delta c} \Delta c_{t+1} + \varepsilon^i_{t+1} \;\; t = -1, 2, ...T \text{ for each } i$$

ii. Average returns should be linearly related to betas,

$$E(R^i) = R^f + \beta_{i,\Delta c} \lambda_{\Delta c}$$

β is the *right hand variable* (x), λ is the *slope coefficient* (β)



iii. i in R^i to emphasize that *This is about why average returns of one asset are higher than of another (cross section). NOT about fluctuation in ex-post return (why did the market go up yesterday?) or predicting returns (will the market go up tomorrow?)*

iv. ER^i, is the reward, β^i the "quantity of risk" varies across assets i. λ is common to all assets i, it is the "price of risk."

5. Intuition and classic theorems.

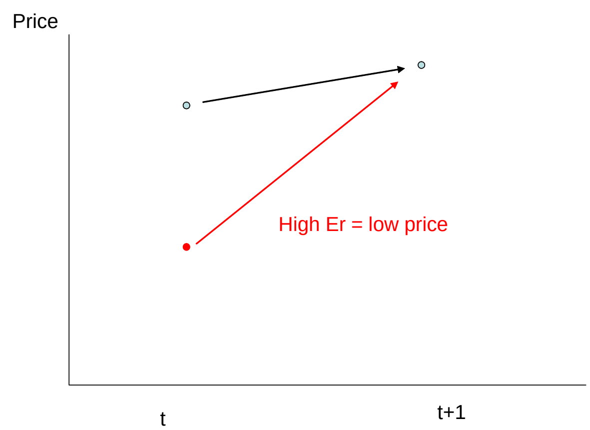
(a) Is high $E(R^e)$ good or bad?

i. Neither. An asset must offer high $E(R^e)$ (good) to compensate investors for high risk (bad).

ii. This is about *equilibrium*, *after* the market has settled down, *after* everyone has made all their trades. It's about $E(R)$ that *will last*, not disappear as soon as investors spot it.

iii. Example: what if we all want to short? The price must fall until we're happy to hold the market portfolio again. How must price and $E(R)$ adjust so that people are happy to hold assets?

(b) Assets that covary *negatively* with m, hence *positively* with consumption growth must pay a higher average return. High $E(R^e) \leftrightarrow$ low price so it's the same intuition as before.



(c) Given volatility (price must go up or down at some point), price (risk-discount) depends on *when* good/bad performance comes. *Average returns are high if beta on* m *or* Δc *is large. Stocks must pay high returns if they tend to go down in bad times.*

i. Price is depressed if a payoff is low in bad times, when "hungry" (high m, low Δc) – High $E(R^e)$.

ii. Price is high if a payoff is high in bad times – Low $E(R^e)$.

(d) Higher γ implies larger price effects. $\gamma =$ *coefficient of risk aversion.*

(e) *Variance* $\sigma(R^e)$ *of an individual asset does not matter, only its covariance with* m *(e.g. consumption growth) matters.* First of many totally (initially) counterintuitive theorems of finance!

(f) "Only systematic risk matters"

$$R^{ei} = \beta_{i.m} m + \varepsilon^i$$

$$var(R^{ei}) = \beta_{i.m}^2 \sigma_m^2 + \sigma_{\varepsilon^i}^2$$

The second component of variance has no effect on mean returns.

(g) *Why is this counterintuitive? This holds *after* you have taken as much as you want, and you adjust c and m! You think of a "marginal" change, buying one more share. Variance *does* matter to the prospective value of a big bite.

(h) Note $E(R^i) < R^f$ is possible! "Insurance"

(i) Note in the consumption model, the market price of risk λ is higher if a) risk aversion is higher or b) if macro volatility is higher.

(j) This is the beginning of all asset pricing models. We just have to think about why we'd use rmrf, hml, smb, etc. in place of Δc.

6. Mean-variance frontier

$$\begin{aligned} E(mR^e) &= 0 \\ E(m)E(R^e) &= -cov(m, R^e) \\ E(R^e) &= -\frac{\sigma(m)\sigma(R^e)\rho_{m,R^e}}{E(m)} \\ \frac{E(R^e)}{\sigma(R^e)} &= -\frac{\sigma(m)}{E(m)}\rho_{m,R^e} \\ \frac{\|E(R^e)\|}{\sigma(R^e)} &\leq \frac{\sigma(m)}{E(m)} \approx \gamma\sigma(\Delta c) \end{aligned}$$

(a) The mean and standard deviation of all asset excess returns must lie inside a cone-shaped region. (These are excess returns, so 0 always exists. The familiar hyperbola is the mean-variance frontier of returns. That's just as easy, but not worth the extra algebra especially when there is no risk free rate.)

(b) The slope of the mean-variance frontier – the reward for taking risk – is higher if *macroeconomic risk* is higher or if *risk aversion* is higher.

(c) No Sharpe ratio can be larger than $\gamma\sigma(\Delta c)$. This justifies the Sharpe ratio limit of the APT.

(d) All assets on the mean-variance frontier are perfectly correlated with m and with each other.

(e) All assets on the mean-variance frontier can be spanned by any two assets on the frontier. "Two-fund" theorem. (Most often, R^f and R^T "tangency portfolio")

7. *Roll theorem: A one-factor pricing model works if and only if the reference return is on the mean-variance frontier*

$$E(R^e) = \beta_{R^e,R^{mv}}\lambda_{mv} \leftrightarrow R^{emv} \text{ is on the mvf.}$$

(a) Proof: $\rho_{m,R^{emv}} = 1$ means R^{emv} is on the mean-variance frontier, and it also means

$$m = a + bR^{emv}$$

Now use the linear factor models trick.

(b) Direct proof

i. A: Mean-variance frontier.

$$\min \sigma^2(R^{ep}) \text{ s.t. } E(R^{ep}) = \mu;$$

$$R^{ep} = w'R^e; E(R^e) = \mu; cov(R^e) = \Sigma$$

$$\min w'\Sigma w \text{ s.t. } w'E = \mu$$

$$\Sigma w = \lambda E$$

$$w = \lambda\Sigma^{-1}E;\; R^{ep}_{t+1} = \lambda E'\Sigma^{-1}R^e_{t+1}$$

This is a formula you will see over and over. λ is a constant, as you can go up or down the mean variance frontier. So a return is mean-variance efficient iff it is a portfolio with these weights

ii. B: to beta models. β is covariance over variance, so the beta of all assets R^e with any portfolio $R^{ep} = w'R^e$ is

$$\begin{aligned}
\beta &= cov(R^{ep})^{-1} cov(R^e, R^{ep}) \\
&= \left[w' cov\left(R^e R^{e\prime}\right) w\right]^{-1} cov\left(R^e R^{e\prime}\right) w \\
&= \left[w'\Sigma w\right]^{-1} \Sigma w
\end{aligned}$$

Now, if – and only if – we choose $w = \lambda \Sigma^{-1} E$, then

$$\begin{aligned}
\beta &= \left[\lambda E'\Sigma^{-1}\Sigma\Sigma^{-1}E\lambda\right]^{-1} \Sigma\Sigma^{-1}E\lambda \\
&= \left[\lambda E'\Sigma E\lambda - 1\right]^{-1} E\lambda
\end{aligned}$$

$$\begin{aligned}
E &= \beta\left[E'\Sigma^{-1}E\lambda\right] \\
E &= \beta\left[w'E\right] \\
E &= \beta E(R^{ep})
\end{aligned}$$

expected excess returns are linear in beta on the portfolio R^{ep}, with $E(R^{ep})$ as factor risk premium, if and only if R^{ep} is mean-variance efficient.

(c) *If you can find a return on the mean-variance efficient portfolio you can price any asset. This statement does not assume returns are normal, and applies to any asset – stocks, bonds, options, fx etc.*

(d) This statement is true even when a multifactor model is true and the CAPM is not true. In that case, the market is not on the MVF (hml beats the market), but a portfolio of rmrf, hml, and smb is on the MVF.

(e) *Not (yet)*: The return R^{emv} = the market portfolio. For example, FF say the 3 factor model means some portfolio with market *and hml and smb* is on the frontier. To get the CAPM we have to argue that the market is on the frontier.

(f) *Not (yet):* Any investor wants to hold a return on the MVF. Multifactor models are all about investors giving up on MV for other objectives. There still *is* a MV frontier, even if people don't want to hold assets on the frontier, and a frontier asset *does* price all the others.

8. Predictable returns? (Why do D/P regressions work? Why does $E_t(R^e_{t+1})$ vary over time?)

$$\begin{aligned}
E_t(R^e_{t+1}) &\approx \gamma cov_t(R^e_{t+1}, \Delta c_{t+1}) \\
&\approx \gamma \sigma_t(R^e_{t+1})\sigma_t(\Delta c_{t+1})\rho_t(R, \Delta c)
\end{aligned}$$

(a) Expected returns may vary over time if risk $\sigma_t(\Delta c_{t+1})$, $\sigma_t(R_{t+1})$ or risk aversion (γ) vary over time.

(b) Let's look at the Sharpe ratio

$$\begin{aligned}
\frac{E_t(R^e_{t+1})}{\sigma_t(R_{t+1})} &\approx \gamma cov_t(R^e_{t+1}, \Delta c_{t+1}) \\
&\approx \gamma \sigma_t(\Delta c_{t+1})\rho_t(R^e, \Delta c)
\end{aligned}$$

(c) Can $\gamma, \sigma_t(\Delta c_t)$ vary day to day? Not plausible, which is why "efficient markets" looks dimly at high frequency trading.

(d) Can γ, $\sigma_t(\Delta c_t)$ vary with business cycles and longer? Possibly! The bottom of a recession has high $\sigma_t(\Delta c_{t+1})$, high γ. This is why D/P regressions do not imply "inefficiency."

(e) Why does γ tend to rise in a recession? A utility function $u(c) = \frac{1}{1-\gamma}(c-x)^{1-\gamma}$ has this property. x can represent leverage, or a habitual level of consumption below which you do not want to fall. As c falls to x, the risk aversion coefficient based on u'' (not γ) rises. Campbell and Cochrane "by force of habit" account for lots of puzzles this way.

9. *Long lived securities.

$$U = u(c_t) + \beta E_t u(c_{t+1}) + \beta^2 E_t u(c_{t+2}) + ... = E_t \sum_{j=1}^{\infty} \beta^j u(c_{t+j})$$

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} = E_t \sum_{j=1}^{\infty} m_{t,t+j} d_{t+j}$$

11.5 Consumption model and the theory of finance

1. Why don't we use Government total nondurable consumption data + power utility function?

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

Or the linearized version $m_{t+1} \approx 1 - \delta - \gamma \Delta c_{t+1}$ leading to

$$\begin{aligned} E(R^{ei}) &= \gamma cov(R^{ei}, m) \\ &= \beta_{i,\Delta c} \times \lambda_{\Delta c} \end{aligned}$$

2. Answer: this is what academics use, to understand where the market, hml, smb, premiums come from ($E(R^{ei}) = \beta_i E(R^{em})$ doesn't address why $E(R^{em})$ is what it is!) But it is not suited for practical use. For example, think of the FF sales growth anomaly. It's much more practical to say sales growth is "explained" by hml betas, and then leave to consumption models the explanation of the hml premium. Consumption doesn't work precisely enough for routine risk and return calculations that don't really care about "rational" vs. "irrational" or "deep economic explanation." Most of the time you just want to know "is this risk I can get somewhere else cheaper?" *Use the right model for the question you want to ask.*

3. Also it doesn't work that well, so the separation between ad-hoc but practical finance models and deep economic explanation makes sense.

   (a) Equity premium puzzle (book)

   (b) Book: FF 25

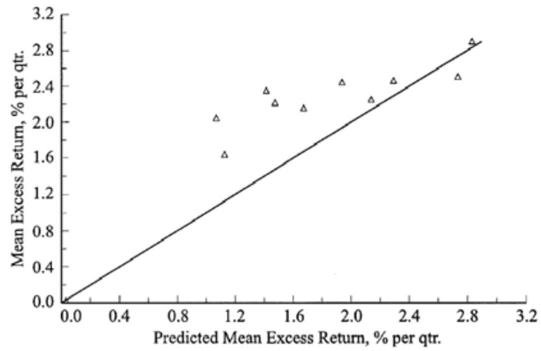


Figure 2.4. *Mean excess returns of 10 CRSP size portfolios versus predictions of the power utility consumption-based model. The predictions are generated by $-R^f \operatorname{cov}(m, R^i)$ with $m = \beta(c_{t+1}/c_t)^{-\gamma}$. $\beta = 0.98$ and $\gamma = 241$ are picked by first-stage GMM to minimize the sum of squared pricing errors (deviation from 45° line). Source: Cochrane (1996).*

(c) Recent research: Maybe it's not so bad after all? Jagannathan and Wang 2005 "Consumption Risk and the Cost of Equity Capital"

Figure 1: Annual Excess Returns and Consumption Betas

Plot figure of average annual excess returns on Fama-French 25 portfolios and their consumption betas. Each two digit number represents one portfolio. The first digit refers to the size quintiles (1 smallest, 5 largest), and the second digit refers to the book-to-market quintiles (1 lowest, 5 highest). Annual excess returns and consumption betas are reported in previous table.

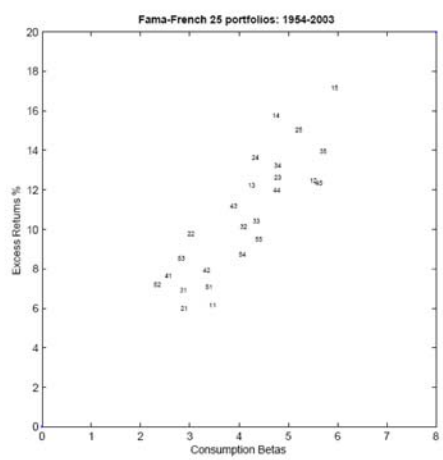
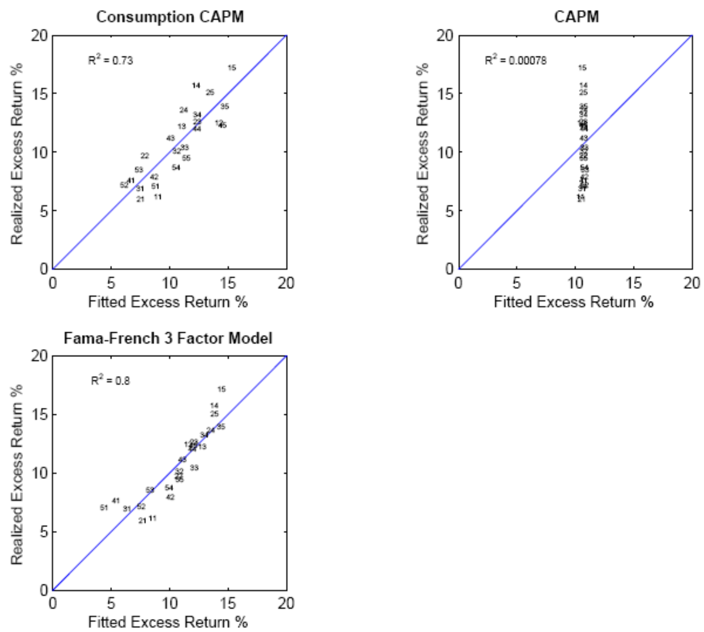


Figure 2: Realized vs. Fitted Excess Returns: FF25 Portfolios

This figure compares realized returns and fitted returns of Fama-French 25 portfolios 1954-2003. Each two digit number represents one portfolio. The first digit refers to the size quintiles (1 smallest, 5 largest), and the second digit refers to the book-to-market quintiles (1 lowest, 5 highest). Three models are compared: CCAPM, CAPM and Fama-French 3 factor model. Models are estimated by using Fama-MacBeth cross-sectional regression procedure. Estimation results are reported in previous table.



(d) However, this result isn't perfect either.

i. It uses the linear approximation $m_{t+1} = 1 - \delta - \gamma \Delta c_{t+1}$ not the real $m = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$

ii. It needs a risk aversion[6] of 31! It doesn't solve the equity premium, and nonlinearities matter a lot for $\gamma = 31$

iii. This only helps on the difference between stock and bond returns. It doesn't do a good job on the level of the risk free rate.

iv. It only works in annual data, only Q4-Q4. (Christmas?)

(e) JC View: This is a very useful result, giving us some hope that consumption is the underlying explanation for returns! Yahoo! But it will not displace factor models for practice soon. If you want to risk-adjust monthly returns, you need something more *practical*

[6] "Consider the slope coefficient, λ_1 in the cross sectional regression equation given by:

$$R_{i,t+4} = \lambda_0 + \lambda_1 \beta_{i,\Delta c} + \varepsilon_{i,t+4}$$

If the standard consumption-based asset pricing model holds, the intercept, $\lambda_0 = 0$ and the slope coefficient, $\lambda_1 = \gamma\ var(\Delta c_{t+4}) / [1 - \gamma[E(\Delta c_{t+4}) - 1]]$, where γ denotes the coefficient of relative risk aversion. The estimated slope coefficient, $\hat{\lambda}_1 = 2.56$, therefore corresponds to an implied coefficient of relative risk aversion of 31." (p. 10-11)

(f) Academic research: Lots going on to find a better utility function, consumption data, etc. I think there is good hope. But it will never be the right model for risk-adjusting a high-frequently anomaly.

4. So, a direct measure of $u'(c)$ is not working well. What to do? How can we use this model as inspiration for something practical?

   (a) Idea 1: ("Absolute pricing") Find other *proxies, data sources* for consumption, marginal utility (CAPM, ICAPM, multifactor models).

   (b) Idea 2: ("Relative pricing") Find discount factors that price one set of assets *by construction.* Don't ask why those assets are priced right, but use them to price other things. (APT, Black-Scholes, Term structure).

11.6 CAPM and Multifactor models

1. Big picture: We want to understand the foundations of the CAPM and FF3F model. Beyond APT, can these models work when R^2 is low? Are there economic rationales behind $E(rmrf)E(hml)$ etc.? "State variables for investment opportunities?"

$$\begin{aligned} E(R^{ei}) &= \beta_{i,R^m}\lambda_{R^m} \\ E(R^{ei}) &= b_i\lambda_{rmrf} + h_i\lambda_{mhl} + s_i\lambda_{smb} \end{aligned}$$

2. Math: is minimal.

   (a) From

$$E(R^{ei}) = R^f cov(R^{ei}, m) = \beta_{R^{ei},m}\lambda_m$$

   we find reasons to say

$$m = a - b \times f$$

   to get

$$E(R^{ei}) = \beta_{R^{ei},f}\lambda_f$$

   Algebra:

$$E(R^{ei}) = R^f cov(R^{ei}, f \times b) = R^f cov(R^{ei}, f) \times b = \frac{cov(R^{ei}, f)}{var(f)} \times \left[R^f b \; var(f)\right]$$

   (b) Intuition: "if low f indicates bad times, when people are hungry, then assets which pay off badly in times of low f must have low prices and deliver high expected returns."

   (c) Example 1, consumption indicates bad times.

$$m_{t+1} = 1 - \delta - \gamma\Delta c_{t+1} \Leftrightarrow E(R^{ei}) = \beta_{i,\Delta c}\lambda_{\Delta c}.$$

   (d) Example 2, CAPM, bad market = bad times, $f = R^{em}$

$$m_{t+1} = a - bR^{em}_{t+1} \;\Leftrightarrow E(R^{ei}_{t+1}) = \beta_{R^{ei}_{t+1},R^{em}_{t+1}}\lambda$$

   (e) Example 3, Roll theorem, $f = R^{emv}$

$$m_{t+1} = a - bR^{emv}_{t+1} \;\Leftrightarrow E(R^{ei}_{t+1}) = \beta_{R^{ei}_{t+1},R^{emv}_{t+1}}\lambda$$

(f) Example 4. The same algebra goes through with multiple factors.

$$\begin{aligned} m_{t+1} &= a - b_1 f_{1t+1} - b_2 f_{2t+1} \Leftrightarrow E_t(R^e_{t+1}) = \beta_{R^e,f_1}\lambda_1 + \beta_{R^e,f_2}\lambda_2 \\ m_{t+1} &= a - b' f_{t+1} \Leftrightarrow E_t(R^e_{t+1}) = \beta_{R^e,f'}\lambda \end{aligned}$$

(g) Issue: What do we get to use for f? "Derivation:" find a story for $m = a - b'f$ and you're done. For example, CAPM: What assumptions do we make to substitute R^m_{t+1} for Δc_{t+1}? The first half of the class will save us a lot of algebra!

3. A precise CAPM derivation, so you can get one glimpse at the sausage factory of asset pricing theory. We need $m = a - bR^W_{t+1}$, and *only* that factor. R^W = return on the wealth portfolio. So we need to tie c to wealth:

   - Suppose people live 2 periods have no job and live off their portfolio.

   $$c_{t+1} = W_{t+1}$$

   $$W_{t+1} = R^W_{t+1}(W_t - c_t)$$

   We need a linear function:

   - Suppose utility is quadratic.

   $$u(c) = -\frac{1}{2}(c^* - c)^2 \Rightarrow u'(c) = c^* - c$$

   - Then

   $$\begin{aligned} m_{t+1} &= \beta\frac{u'(c_{t+1})}{u'(c_t)} = \beta\frac{(c^* - c_{t+1})}{(c^* - c_t)} \\ &= \beta\frac{\left(c^* - R^W_{t+1}(W_t - c_t)\right)}{(c^* - c_t)} \\ &= \frac{\beta c^*}{(c^* - c_t)} - \frac{\beta(W_t - c_t)}{(c^* - c_t)}R^W_{t+1} \\ m_{t+1} &= a_t - b_t R^W_{t+1} \end{aligned}$$

   We have "derived the CAPM."

4. *Another "CAPM derivation." Assume log utility, but infinite life. With log utility the wealth portfolio claim is

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} c_{t+j}$$

$$\begin{aligned} \frac{p_t}{c_t} &= E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} \left(\frac{c_{t+j}}{c_t}\right) \\ \gamma &= 1 : \frac{p_t}{c_t} = E_t \sum_{j=1}^{\infty} \beta^j = \frac{\beta}{1-\beta}. \end{aligned}$$

Then, we want to tie m to R^W

$$\begin{aligned} R^w_{t+1} &= \frac{p_{t+1}+c_{t+1}}{p_t} = \frac{\left(\frac{p_{t+1}}{c_{t+1}}+1\right)\frac{c_{t+1}}{c_t}}{\frac{p_t}{c_t}} = \frac{\frac{\beta}{1-\beta}+1}{\frac{\beta}{1-\beta}}\frac{c_{t+1}}{c_t} \\ &= \frac{\beta+1-\beta}{\beta}\frac{c_{t+1}}{c_t} = \frac{1}{\beta}\frac{c_{t+1}}{c_t}\left(=\frac{1}{m_{t+1}}\right) \\ m_{t+1} &= 1/R^W_{t+1} \approx a - bR^W_{t+1} \end{aligned}$$

Thus, we can replace R^w_{t+1} for Δc_{t+1}.

5. Comments on CAPM derivations

   (a) CAPM $\Leftrightarrow$ The market portfolio is on the mean-variance frontier $\Leftrightarrow$ Investors want to hold mean-variance efficient portfolios.

   (b) Note this *is* the consumption model. We just substituted a *determinant* of consumption (market return) for consumption itself.

   (c) Does this seem Artificial? Yes. Why? Look at the quadratic derivation: 1) Two periods to make the market return the *only* determinant of consumption. *The key to a model is not "this is a plausible risk factor, but this is ALL the risk factors.* 2) Quadratic to make m *linear* in the market return. That's minor and there are lots of better ways to get it.

   (d) OK, we derive the CAPM, but note the implicit (and soon forgotten) predictions about consumption. For example, in the lognormal derivation

      i. $c_{t+1}/c_t = \beta R^W_{t+1}$ Consumption growth tracks market returns perfectly! $\sigma(\Delta c) = 18\%$, $corr(\Delta c, R^m) = 1$?! It implies consumption is extremely volatile or market returns are not

      ii. p/c is a constant??

   (e) The assumptions are extreme. OK, so we've been waiting for years for practical alternatives to CAPM. It was not there by deep assumptions, it was there because it worked so well. Now that we look at the real theory, we should not be at all surprised that it doesn't work so well. What's amazing is that it took so many years to find practical multi-factors!

   (f) Multifactor model derivations follow the same idea. They find "assumptions" to substitute out for Δc_{t+1} Since noone checks these assumptions in practice, I see no need to drag you through them.

   (g) I derive the CAPM as a special case of the consumption model, when people have no job or outside income, live two periods or the world is iid and log utility. Be aware of the conventional derivation: People want a mean-variance portfolio; each security contributes to portfolio variance in proportion to beta.

6. Inspiration for multifactor models. What variables f other than consumption itself might indicate bad times – high marginal utility; hunger?

$$m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)} \approx a - b'f_{t+1}$$

Think of the linear m as a local (Taylor) approximation, just as we already did like

$$\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \approx 1 - \delta - \gamma \Delta c_{t+1}.$$

(a) How about the *determinants* of consumption: wealth, income, news?

(b) **What can be** f? Low market return = low wealth, low consumption. CAPM

$$m_{t+1} = a - bR^m_{t+1}$$

Equivalently,

$$E(R^i) - R^f = \beta_{i,R^m} \times \left[E(R^m) - R^f\right]$$

(Notation: R^m is traditional for "market." It has nothing to do with the discount factor m.)

(c) **What can be** f? (My favorite). *Labor / proprietary income.* Direct: lost job? Consumption will decline. Indirect: Don't want stocks to decline when you just lost your job. Other macroeconomic variables. Investment, GDP, interest rates, unemployment, inflation, etc. have all been used. Motivation: They affect consumption

(d) **What can be** f? *News* about future investment opportunities. ("ICAPM") News that $E_{t+1}(R_{t+2})$ is low is bad news for a long-lived investor; long run wealth will now be lower. Consumption goes down when $E_t(R_{t+1})$ goes down. Indirect: We want assets that pay well when this happens. ICAPM

$$\begin{aligned} m_{t+1} &= a - bR^m_{t+1} - b_2 f_{t+1} \\ E(R^i) - R^f &= \beta_{i,R^m} E(R^m - R^f) + \beta_{i,f}\lambda_f \end{aligned}$$

Examples: Changes in D/P, interest rates. (Note: this story requires $\gamma \neq 1$, as per problem set question, and the sign of λ_f depends on $\gamma >< 1$).

(e) **What can be** f? Most of this is pretty discouraging. We wanted something more practical than consumption, and here we are with stuff even harder to measure. Where do portfolios like FF3F come from? A: Mimicking portfolios, so you don't need data on labor income, news, macro variables. *The portfolio of assets formed by a regression of any m on returns is also a discount factor.*

i. Why? Think of running a regression of m on all returns,

$$m = a + b'R^e + \varepsilon = a + \sum_i b_i R^{ei} + \varepsilon$$

By construction,

$$E(\varepsilon R^e) = 0$$

Thus,

$$\begin{aligned} 0 &= E(mR^e) \\ 0 &= E\left[\left(a + b'R^e + \varepsilon\right) R^e\right] \\ 0 &= E\left[\left(a + b'R^e\right) R^e\right] \end{aligned}$$

but

$$R^{ep} = b'R^e$$

is of course a portfolio. So, by our linear factor trick,

$$E(R^{ei}) = \beta_{i,R^{ep}} E\left(R^{ep}\right)$$

Thus, an excuse for portfolios as risk factors.

ii. Bigger picture, and we'll do it again and again: *the right hand side of a regression is a portfolio.* We can just run regressions to construct the "optimal hedge"!

iii. Fama and French say hml and smb are "Mimicking portfolios for state variables of concern to investors." This is what they mean! Note most people (like FF) cite this, but *do not* write what fundamental m, variables, they have in mind nor do they run the regression.

(f) **What can be** f? Back to the APT. Our theorems all work backwards too, so if alphas are zero in the APT expression of mean returns,

$$E(R^{ei}) = \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2 + ...$$

were betas are defined by

$$R_t^{ei} = \alpha_i + \beta_{i,1} f_{1t} + \beta_{i,2} f_{2t} + .. + \varepsilon_{it}$$

then we also can express pricing as

$$\begin{aligned} 0 &= E(m_{t+1} R_{t+1}^e) \\ m_{t+1} &= a - b_1 f_{it+1} - b_2 f_{2t+1} ... \end{aligned}$$

(g) The real question is **what can't be f?** That's the heart of a derivation. The CAPM says not just "the market matters" but "the market is the *only* thing that matters." *This* is the key to "deriving the CAPM." We need a story for

$$m = \beta \frac{u'(c_{t+1})}{u'(c_t)} = a - b R_{t+1}^W$$

and *only* that factor – nothing else but market returns drives consumption. The actual assumptions you need to get that are pretty unrealistic: nobody has a job, and either people live only two periods, have quadratic utility, or returns are not correlated over time.

(h) The derivations make it clear that the f has somehow to be related to aggregate good and bad times, consumption, the behavior of wide swaths of asset returns, macroeconomic events, etc. "Factors" that have nothing to do with the above shouldn't matter.

(i) But... you see the derivations are so loose you can get away with a lot. Empirically people don't pay much attention to derivations. They take inspiration from these stories and see what works.

7. I emphasized link to consumption, e.g. f goes down implies consumption goes down. It's also important to understand the *Portfolio logic for multifactor models:* Why should something more than market return show up as a factor? (This mirrors the standard portfolio logic for the CAPM.)

(a) Example:

i. Stocks A,B have the same mean, variance, beta. In a recession (bad times, bad consumption), for given level of the market return, A goes up while B goes down.
ii. According to CAPM should you care?
iii. Do *you* want A or B?
iv. ⇒People want more A ⇒ Price of A goes up ⇒ Expected returns of A go down.
v. ⇒People want less B ⇒ Price of B goes down ⇒ Expected returns of B go up.
vi. ⇒Expected returns depend on recession sensitivity as well as market sensitivity. In equations,

$$E\left(R^{ei}\right) = \beta_{i,m}\lambda_m + \beta_{i,\text{recess}}\lambda_{\text{recess}}$$

vii. Equivalent to a "formal" derivation in which people have outside income, privately held businesses, or news about the future.
viii. "ICAPM." A, B have the same mean, variance, beta. But when news comes that market returns in the future will be bad, A goes up and B goes down. .... The tendency to move against such news is a priced factor.
ix. Bond example for ICAPM. Why do people hold long term bonds, with the same $E(R)$ as short term bonds but huge standard deviation? Answer: When interest rates go down, meaning bad returns in the future, long term bond prices rise. Long term bonds are less risky for a long-term investor than short-term bonds. The puzzle is the opposite – long term bonds should have a *lower* expected return than short term bonds! The tendency of an asset to rise when interest rates decline –beta on interest rates as a state variable – should be highly valued, and expected returns should be lower for assets with that beta.

(b) Important facts

i. λ can be negative. λ typically is negative for "bad" factors like oil price rise, positive for "good" factors like rise in market, consumption growth.
ii. The size of λ is determined by how much the *average* investor wants to avoid that risk; larger for more important risks and higher risk aversion. "Marginal" investor is a misnomer.
iii. It must be *aggregate* risk to affect prices. Risks that as many people like as dislike are merely transferred through assets. Oil sensitive stocks: Texas sells, we buy, no price effect. If a risk only matters to *you*, the average investor does not shy away from it, assets that covary with it do not get "underpriced" and no expected return arises. For example, a value effect requires more people who lose if value firms go down than growth, or it all offsets.

8. *Alternative representations. There are lots of ways to write any model!

(a) *Every multifactor model can be written as a single factor model.* Example

$$\begin{aligned}
E(R^{ei}) &= \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 \\
&\leftrightarrow \\
0 &= E(mR^e);\ m = a - b_1 f_1 - b_2 f_2 \\
0 &= E(mR^e);\ m = a - (b_1 f_1 + b_2 f_2) \\
&\leftrightarrow \\
E(R^{ei}) &= \beta_{im}\lambda_m
\end{aligned}$$

Notice m is a *single special* combination of f_1 and f_2

(b) Related: Recall (35000, fun fact above class): that *there is always some portfolio R^{emv} on the mean-variance frontier,* and that *it is always true that* $E(R^{ei}) = \beta_{i,R^{emv}} E(R^{emv})$ – a single factor representation using a mean-variance efficient portfolio.

Thus, for example, if the FF3F model is right, there is a combination of rmrf, hml, and smb that is on the mean-variance frontier, and that new portfolio could act as a single factor model.

$$\begin{aligned}
E(R^{ei}) &= b_i E(rmrf) + h_i E(hml) + s_i E(smb) \leftrightarrow \\
R^{emv} &= a \times rmrf + b \times hml + c \times smb \leftrightarrow \\
E(R^{ei}) &= \beta_{i,mv} E(R^{emv})
\end{aligned}$$

(c) Aha, but how do you find a portfolio on the mean-variance frontier? We can cast all our "theories" as statements about how to find such a portfolio.

(d) The CAPM says rmrf is on the mean variance frontier. Thus, multifactor models say rmrf is *not* on the mean-variance frontier. The average investor gives up some mean/variance to get a portfolio that (say) does not fall so much in recessions.

But there still is some other portfolio on the frontier – $rmrf$ plus a bit of hml and smb in the FF model. Again, we get a single factor model with $R^{mv} = rmrf + \gamma_1 hml + \gamma_2 smb$ This model gives the same α but different (less interpretable?) betas

(e) Why one representation vs. another? *Use whatever gives more intuition.* For FF3F, looking at separate size and b/m betas is interesting. You could express the exact same result as a single-beta model with one combination of rmrf, smb, hml, but you'd lose intuition. You'd lose "this one is like value" "this one is like small stocks" although α would be the same.

9. Comparing APT with CAPM and factor models.

(a) The APT is a multifactor model. But f *are portfolios that do a good job of explaining cross-correlation of asset returns,* good R^2 in

$$R^{ei}_{t+1} = \alpha_i + \beta_{i1} f_{1,t+1} + \beta_{i,2} f_{2,t+1} + ... + \varepsilon^i_{t+1}, T = 1, 2, ...T \text{ for each } i$$

not necessarily "proxies for state variables."

(b) High R^2 in the time series regression *implies* a factor model (APT logic). High R^2 in the time series regression is not *required* for a factor model. Other stories ("state variables") do not need high R^2.

(c) Example. Is the CAPM a factor model or an APT? The real CAPM is a general model that should hold for any R^2. The CAPM is also an APT, that under much milder assumptions should still hold for assets (index futures, say) that have very high R^2 when regressed on the market portfolio.

(d) CAPM vs. APT. The APT says all the covariance factors f are *candidates* for pricing. But it does not say all of them *have* to matter for pricing. For example, suppose f^1 is the market and f^2 is hml, and

$$\begin{aligned}
R^{ei}_t &= [\alpha_i = 0] + \beta_{i1} f^1_t + \varepsilon_{it}; \quad R^2 = 0.2 \\
R^{ei}_t &= [\alpha_i = 0] + \beta_{i1} f^1_t + \beta_{i2} f^2_t + \varepsilon_{it}; \quad R^2 = 0.99
\end{aligned}$$

APT logic says we should see

$$E\left(R_t^{ei}\right) = [\alpha_i = 0] + \beta_{i1} E\left(f_t^1\right) + \beta_{i2} E\left(f_t^2\right)$$

a two factor model. But it does NOT *require* that $E(f_t^2) \neq 0$.

(e) APT and CAPM. In this situation,

$$R_{t+1}^{ei} = \alpha_i + \beta_{i1} f_{t+1}^1 + \beta_{i2} f_{t+1}^2 + \varepsilon_{t+1}^i.$$

If $\alpha_i = 0$,

$$E(R^{ei}) = \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 = \beta_{i1} E(f^1) + \beta_{i2} E(f^2)$$

The CAPM is the case that $f^1 = rmrf$, and $E(f^2) = 0$. *The CAPM is perfectly fine with the idea that additional factors help to explain the variance of returns.* Industry factors are a good example. *But, they should not get any extra expected return.* The central prediction of the CAPM is *not* that β_2 should $= 0$. The central prediction of the CAPM is that $E(f^2)$ should $=0$, so that leaving f^2 out does not generate an α. (More precisely, the CAPM's prediction is that $E(f^2) = \beta_{2,1} E(f^1)$, i.e. $\alpha_2 = 0$. For class discussion, assume that the factors f^1 and f^2 are uncorrelated.)

(f) Example:There are many common risk factors – such as industry portfolios – that are important for risk management, but do not generate a premium. This was perfectly ok by the CAPM. The CAPM is fine with, for example,

$$R_t^{ei} = \beta_{im} R_t^{em} + \gamma_i (R_t^{eI} - \beta_{I,m} R_t^{em}) + \varepsilon_t^i$$

where R^{eI} is an industry portfolio. The CAPM is fine that $\gamma > 0$, and the R^2 of this regression is improved by including industry factors. I made the industry factor orthogonal to market returns here to emphasize that I am explaining R^{ei} by a movement in R^I that is uncorrelated with the market. But as you can see clearly, if the CAPM holds, then

$$\begin{aligned} E\left(R_t^{ei}\right) &= \beta_{im} E\left(R_t^{em}\right) + \gamma_i E(R_t^{eI} - \beta_{I,m} R_t^{em}) \\ &= \beta_{im} E\left(R_t^{em}\right) + \gamma_i 0 \end{aligned}$$

11.7 Asset Pricing Models final comments

What's the right model? *It depends what you want to use it for.*

1. Deep economic explanation, fighting with behavioral guys over "rational" and "irrational" markets? The factors had better be well tied to real macroeconomic risks. Even hml and smb are pretty tenuous here. umd even more so.

2. Evaluation of a manager – Could I have gotten the same average return with some simple style indices? Now the factors should just be the tradeable style indices you could invest in. It doesn't matter if *they* are "right", it only matters if the manager can beat them, and you know how to access them.

3. Risk management/hedging? Models for *mean* are different than models for *variance.* The BARRA model has 67 "factors." Many of them have no premium. You get *means* right if you ignore them. And you don't care about alphas! For these purposes R^2 does matter, and the "state variable" nature of factors does not matter. You want tradeable, hedgeable factors.

4. Seeing if a new expected return strategy is genuinely new, or just new way of getting some known anomaly (value, momentum)? Just like evaluation, it doesn't matter really how pure the factors are.

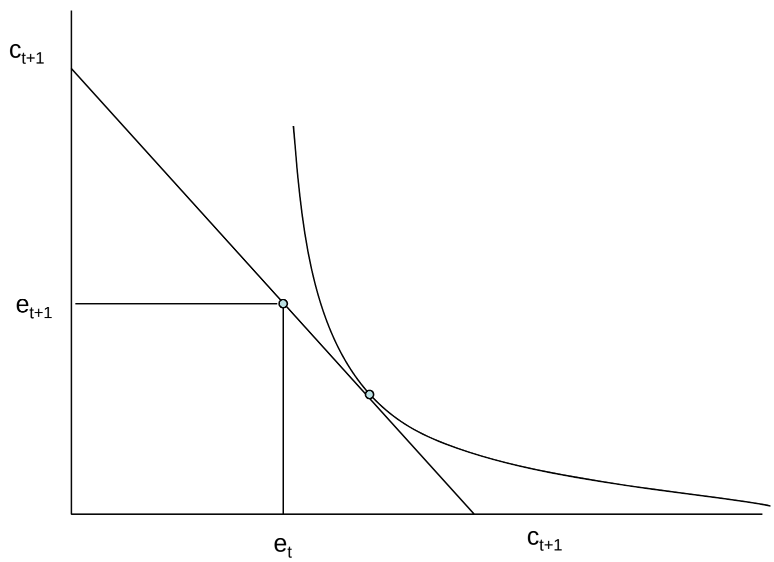
12 *Week 4 Asset Pricing Theory Extras

1. The chicken and the egg. Does consumption determine asset returns, or do asset returns determine consumption?

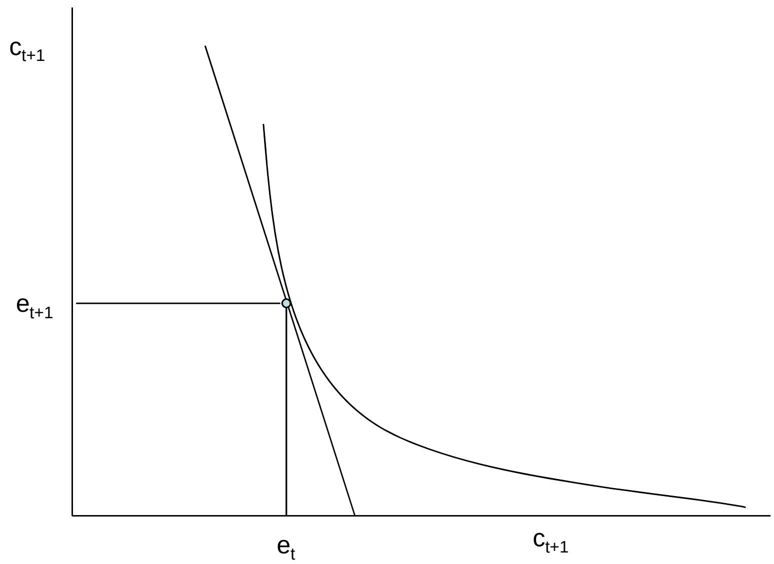
   (a) Answer: in general both. Individual investors see asset prices and payoffs as fixed, and our basic equation determines their consumption. To the economy, however, the equation may determine asset prices given consumption.

   (b) Example 1: "endowment economy." Suppose that our investors live on an island, and eat coconuts which fall from trees. The amount of coconuts c_t and c_{t+1} are thus fixed. Impatient investors may try to borrow coconuts from others, promising to pay back more tomorrow. As they do so, interest rates rise until each investor is happy eating the coconuts that have fallen. Each individual can still borrow from others, so each individual sees the price as fixed and his consumption as the "endogenous" variable. But for the economy as a whole, if a lot of coconuts fall on a given day, then the interest rate will fall until supply equals demand again.

   To be specific, suppose the amount that falls from the tree e_t and e_{t+1} is always the same. The identical impatient islanders wake up, see an interest rate of $R^f = 1.0$ and want to borrow. I graphed e_t, e_{t+1} on the 45 degree line. At the interest rate R^f=1, I graphed each investor's optimum down to the right. This market is not clearing, more pople want to borrow – $c_{t+1}$is higher than e_{t+1} – than want to lend.



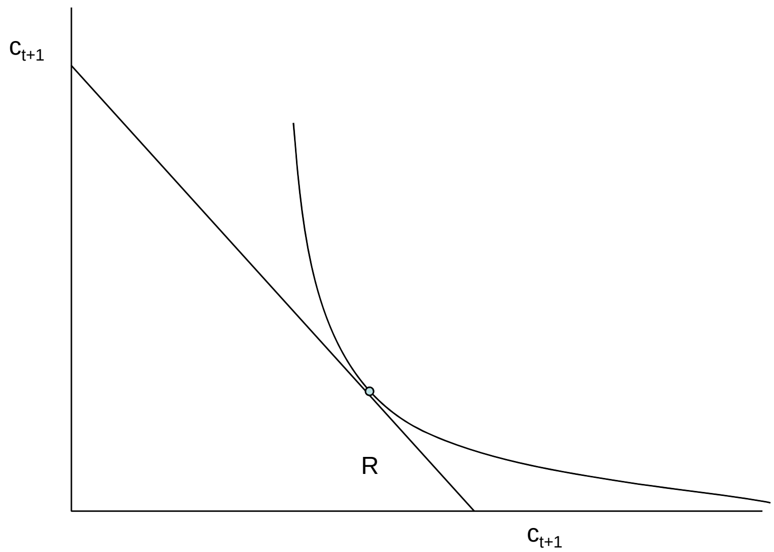
   What happens? The interest rate must rise, and it must keep rising until each of the investors is content to eat just one coconut today and one coconut tomorrow. As here



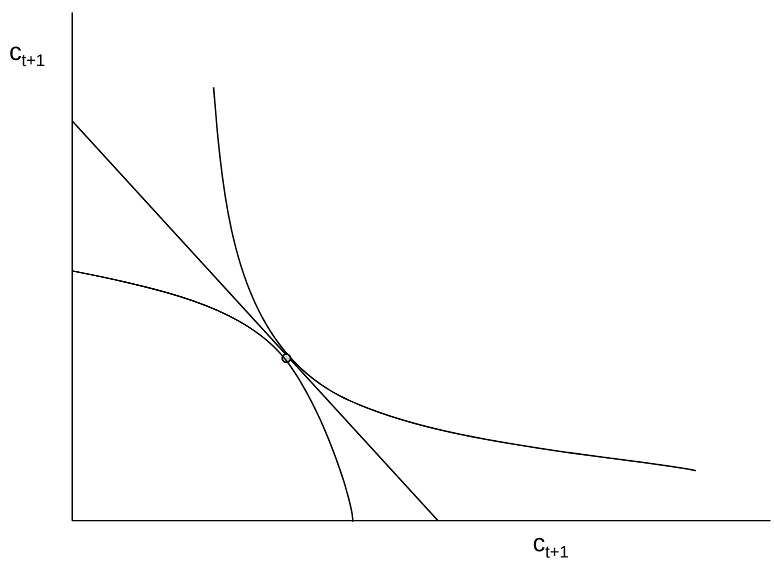
So each investor can borrow and lend all he wants to. He's impatient, but the huge interest rates just offset his impatience and he chooses not to. Interest rates determine consumption at the individual level. From the economy's perspective, though, it's the endowment (consumption) that is driving interest rates. If two coconuts fall today, and one will fall tomorrow, the endowment point shifts to the right and interest rate fall. (Graphing that is a good exercise.)

(c) Example 2. "linear technology." Suppose instead the island has wheat, and each kernel planted gives $R = 2$ kernels the next year. Thus, the interest rate R^f will be two. Each individual still sees a fixed interest rate and our equation determines consumption. Now, if lots of people want to save, however, rather than drive interest rates down, both individuals and the economy as a whole can plant, save, and produce greater consumption in the future. In this case, it is asset returns that are fixed and consumption that adjusts, both for individual and for the economy as a whole.

I graphed this situation next. If the interest rate were anything but R, the *producers* would invest everything or nothing and take the economy to one of the corners. So, the production technology determines the asset return R, and then consumption adjusts, both for the indivdual and for the economy. If the economy discovers a new kind of wheat that delivers 3 seeds tomorrow for every seed planted tonight, the interest rate goes up to $R^f = 3$, and consumption adjusts. That too is a good graph to make as an exercise.



(d) Example 3. Reality. In reality, there is a concave production possibility frontier, between the point of example one and the line of example 2. If people are impatient, and interest rates rise, firms will pull wheat out of planting and make bread with it. They can provide some more wheat today, but then less tomorrow. As here



The bowed line on the right represents the production possibility frontier. The more wheat the farmers sow – moving to the left and up, reducing c_t – the more they get out tomorrow, increasing c_{t+1}. Now the equilibrium consumption and interest rate is determined by the interaction of preferences and technology. Each individual still thinks prices are given and he or she is adjusting consumption.

(e) In sum then, the chicken and egg question comes down to the production technology of the economy. In real life, if more want to save than borrow, there is some more investment and a bit of interest rate rise Once we get to equilibrium, each individual still sees prices as fixed. There is not a clear chicken and egg – our equation describes relationships that hold in equilibrium, but for the economy as a whole there is not a clear separation of consumption determines interest rates or interest rates determine consumption.

2. From $p = E(mx)$ to all of asset pricing. Everything we do is just special cases, that are useful in various circumstances.

   (a) In most of finance we do not use consumption data. We instead use other tricks to come up with an m that works better in practical applications.

   (b) **Theorem:** *If there are no arbitrage opportunities, then we can find an m with which we can represent prices and payoffs by* $p = E(mx)$ Thus, the m structure allows us to do "no arbitrage" asset pricing.

   (c) Bond prices

$$
\begin{aligned}
P_t^{(1)} &= E_t(m_{t+1} \times 1) \\
P_t^{(2)} &= E_t(m_{t+1} m_{t+2} \times 1).
\end{aligned}
$$

   Term structure models (Cox Ingersoll, Ross, etc.): model m_{t+1}, (model Δc_{t+1}),

      i. For example

$$m_{t+1} = \phi m_t + \varepsilon_{t+1}$$

      Then

$$
\begin{aligned}
P_t^{(1)} &= \phi m_t \\
P_t^{(2)} &= \phi^2 m_t = \phi P_t^{(1)} \\
P_t^{(3)} &= \phi^3 m_t = \phi^2 P_t^{(1)} \\
P_t^{(N)} &= \phi^N m_t = \phi^{N-1} P_t^{(1)}
\end{aligned}
$$

      Look! We have a "one-factor arbitrage-free" model of the term structure. We can draw a smooth curve through bond prices (and then yields) in a way that we know does not allow arbitrage. (Week 8)

   (d) Option pricing (Black-Scholes). Rather than price options from consumption, find m that prices stock and bond, then use that m to price option. (*Asset Pricing* derivation of Black-Scholes)

3. "Risk-neutral pricing". How $p = E(mx)$ is the same as what you learned in options/fixed income classes.

   (a) Our formula

$$p = E(mx) = \sum_{s=1}^{S} \pi_s m_s x_s$$

(b) "Risk-neutral probabilities" (Veronesi, options pricing) Define

$$\begin{aligned}
p &= \sum_s \pi_s m_s x_s \\
&= \left(\sum_s \pi_s m_s\right) \sum_s \frac{\pi_s m_s}{\left(\sum_s \pi_s m_s\right)} x_s \\
&= \frac{1}{R^f} \sum_{s=1}^{S} \pi_s^* x_s \\
p &= \frac{1}{R^f} E^*(x)
\end{aligned}$$

if we define

$$R^f \equiv \frac{1}{E(m)} = \frac{1}{\sum_s \pi_s m_s}.$$

and

$$\pi_s^* = \frac{\pi_s m_s}{\sum_s \pi_s m_s} = R^f \pi_s m_s$$

(we'll see $R^f = 1/E(m)$ below; for now just use it as a definition)

i. Note

$$\sum_s \pi_s^* = 1$$

so they could be probabilities[7].

ii. Interpretation of $p = \frac{1}{R^f} E^*(x)$: price equals risk-neutral expected value using special "risk-neutral probabilities" π^*

(c) *A discount factor m is the same thing as a set of "risk neutral probabilities"*

i. Option 1: find "probabilities" π^* that price stock and bond using $p = \frac{1}{R^f} E^*(x)$. Use those probabilities to price option using the same formula

ii. Option 2: find m that prices stock and bond using $p = E(mx)$. Use that m to price option using $p = E(mx)$. (Using true probabilities)

iii. These are exactly the same thing!

4. Long lived securities, the explicit derivation:

$$U = E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

pay $p_t \xi$, get $\xi d_{t+1}, \xi d_{t+2} ...$

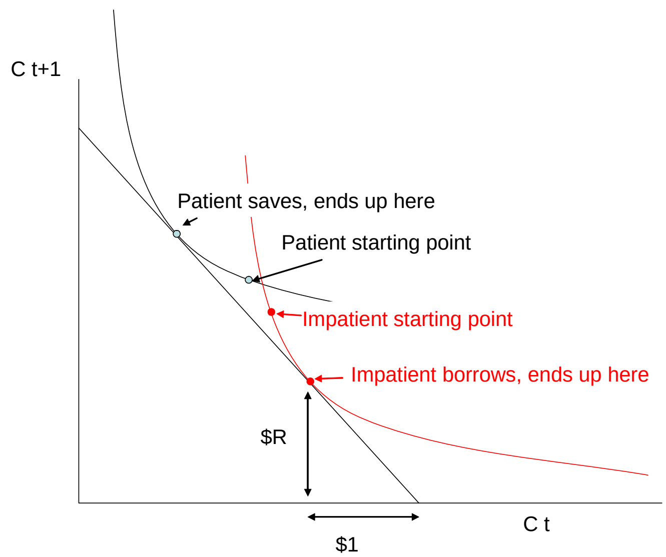
$$\begin{aligned}
p_t u'(c_t) &= E_t \sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) d_{t+j} \\
p_t &= E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \\
p_t &= E_t \sum_{j=1}^{\infty} m_{t,t+j} d_{t+j} = E_t \sum_{j=1}^{\infty} (m_{t+1} m_{t+2} ... m_{t+j}) d_{t+j}
\end{aligned}$$

This is the present value formula with *stochastic discount factor.*

[7] Also since m comes from $u'(c)$ and $u'(c) > 0$, $\pi_s^* > 0$ which probabilities have to obey

5. Question: What if people have different γ, β, or different utilities? Then we get different prices depending on who we ask?

   Answer: Yes if we're asking about genuinely new securities that have not been sold yet. But no if we are talking about market prices, the usual "at equilibrium." In a market everyone adjusts *their consumption and portfolio* until they value things at the margin the same way. Example: One is patient, prefers consumption later. One investor is impatient, prefers consumption now. At their starting point, the patient investor implies a lower interest rate, as you would expect. But facing the same market rate, P saves more and I borrows more, until *at the margin* they are willing to substitute over time at the same interest rate, as shown.



6. Question: you slipped in to talking about economy-wide average consumption, not individual consumption. What's up with that?

   Answer: Right. There is a "theory of aggregation" that lets us do this. Here's what needs to be proved: that the average consumption across people responds to market prices just as if there is a single consumer with "average" risk aversion γ and discount rate β doing the choosing. Under some assumptions, it's true. This is natural – in thinking about "high interest rates got people to save more" we don't obviously have to talk about some people being different than others; we can take first cut at the problem by thinking about the behavior of a "representative person."

7. Multifactor models, the explicit vector version

$$E(R^{ei}) = \beta_{i,f^1}\lambda_1 + \beta_{i,f^2}\lambda_2 + \ldots \text{or } E(R^{ei}) = \beta_i'\lambda$$

   where β are (usually) defined from *multiple* regressions,

$$\begin{aligned} R^{ei}_{t+1} &= \alpha + \beta_{i,f^1} f^1_{t+1} + \beta_{i,f^2} f^2_{t+1} + \ldots + \varepsilon^i_{t+1};\ t = 1, 2, \ldots T \\ \text{or } R^{ei}_{t+1} &= \alpha + \beta_i' f_{t+1} + \varepsilon^i_{t+1} \end{aligned}$$

$$\beta = \begin{bmatrix} \beta_{i,f^1} \\ \beta_{i,f^2} \\ \vdots \\ \beta_{i,f^n} \end{bmatrix};\ f_{t+1} \begin{bmatrix} f^1_{t+1} \\ f^2_{t+1} \\ \vdots \\ f^n_{t+1} \end{bmatrix}$$

8. The real APT with multiple assets.

(a) You can be smarter than finding sharpe ratios of individual opportunities. How about the Sharpe ratio from clever *portfolios*? For example, suppose you find a trading opportunity, but $\sigma(\varepsilon^i)$ is large, so the Sharpe ratio $\alpha^i/\sigma(\varepsilon^i)$ is small. Well, if you can find several of these, then the *portfolio* variance will be less than the sum of *individual* variances. For example, suppose there are two opportunities with the same α and $\sigma(\varepsilon)$. Now, the Sharpe ratio of the portfolio is

$$\frac{\frac{1}{2}\alpha + \frac{1}{2}\alpha}{\sigma\left(\frac{1}{2}\varepsilon^1 + \frac{1}{2}\varepsilon^2\right)}$$

If the errors are uncorrelated that is

$$\frac{\alpha}{\sqrt{\frac{1}{4}\sigma^2\left(\varepsilon^1\right) + \frac{1}{4}\sigma^2\left(\varepsilon^2\right)}} = \frac{\alpha}{\frac{1}{\sqrt{2}}\sigma(\varepsilon)} = \sqrt{2}\frac{\alpha}{\sigma(\varepsilon)}$$

Similarly, with N assets, you would get a Sharpe ratio $\sqrt{N}$ higher. What could go wrong? If the ε contain another factor, so they are all correlated! But if they do not contain another factor, if they really are uncorrelated, then this argument would show we get small α even though *individual* R^2 are not that low.

(b) The real APT, then: The maximum Sharpe ratio available in ever cleverly chosen *portfolio* of many R^{ei} should be small. This is also a useful formula in general: How do you find optimal portfolios!

(c) Background: A set of returns R^e with covariance matrix Ω. What is the portfolio that gives the best Sharpe ratio? Answer:

$$w = \text{constant} \times \Omega^{-1}E(R^e)$$

We will only get an answer up to scale of course, since $2\times R^{ep}$ has the same SR as R^{ep}. What is the SR of the best portfolio? Answer: $SR_{max} = \sqrt{E(R^e)'\Omega^{-1}E(R^e)}$

(d) Derivation (good practice with matrices!)

$$R^e = \begin{bmatrix} R^{e1}_{r+1} \\ R^{e2}_{t+1} \\ \vdots \\ R^{eN}_{t+1} \end{bmatrix}; \; \Omega = \begin{bmatrix} \sigma^2(R^{e1}) & cov(R^{e1}, R^{e2}) & cov(R^{e1}, R^{e3}) & \cdot \\ & \sigma^2(R^{e2}) & cov(R^{e2}R^{e3}) & \cdot \\ & & \sigma^2(R^{e3}) & \cdot \\ & & & \sigma^2(R^{eN}) \end{bmatrix}; w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

Portfolios

$$R^p = \sum_{i=1}^{N} w_i R^{ei} = w'R^e;$$

Problem:

$$\max_{\{w\}} \frac{E(R^p)}{\sigma(R^p)}$$

This is the same as

$$\min_{\{w\}} \sigma^2(R^p) \text{ given } E(R^p) = \mu,$$

the mean-variance frontier.

$$\begin{aligned} \sigma^2(R^p) &= var(w'R^e) = w'\Omega w \\ E(R^p) &= E(w'R^e) = w'E(R^e) \end{aligned}$$

$$\min_{\{w\}} w'\Omega w - \lambda w' E(R^e)$$
$$\begin{aligned} \Omega w &= \lambda E(R^e) \\ w &= \lambda \Omega^{-1} E(R^e) \end{aligned}$$

Thus we have, *Answer 1: Optimal portfolio.*

(e) We were here to find Sharpe ratios,

$$\frac{E(R^p)}{\sigma(R^p)} = \frac{w'E(R^e)}{\sqrt{w'\Omega w}} = \frac{\lambda E(R^e)'\Omega^{-1}E(R^e)}{\sqrt{\lambda^2 E(R^e)'\Omega^{-1}E(R^e)}} = \sqrt{E(R^e)'\Omega^{-1}E(R^e)}$$

Answer 2: Max SR from these assets is $\sqrt{E(R^e)'\Omega^{-1}E(R^e)}$

(f) Now, what about our multifactor model? Start with a regression

$$R^e_{t+1} = \alpha + \beta f_{t+1} + \varepsilon_{t+1}$$

where

$$R^e_{t+1} = \begin{bmatrix} R^{e1}_{r+1} \\ R^{e2}_{t+1} \\ \vdots \\ R^{eN}_{t+1} \end{bmatrix} ; \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} ; \varepsilon_{t+1} = \begin{bmatrix} \varepsilon^1_{r+1} \\ \varepsilon^2_{t+1} \\ \vdots \\ \varepsilon^N_{t+1} \end{bmatrix};$$

i.e. FF 25 test assets, FF 3 factors. Form a portfolio of the assets, possibly hedged with factors.

$$R^{ep}_{t+1} = w'R^e_{t+1} - v'f_{t+1}$$

Solve the problem: max Sharpe ratio of such portfolios

$$\max_{\{w,v\}} \frac{E\left(R^{ep}\right)}{\sigma\left(R^{ep}\right)}$$

Answer:

$$SR^2 = \underbrace{E(f)'cov(f,f')^{-1}E(f)}_{\text{max } SR \text{ from factors alone}} + \underbrace{\alpha' cov(\varepsilon,\varepsilon')^{-1}\alpha}_{\text{extra SR from exploiting } \alpha}$$

Interpretation: You can buy any portfolio of f, and you can get any portfolio of $(\alpha+\varepsilon)$ by buying R^e and selling f. Now, f and ε are uncorrelated so the problem separates.

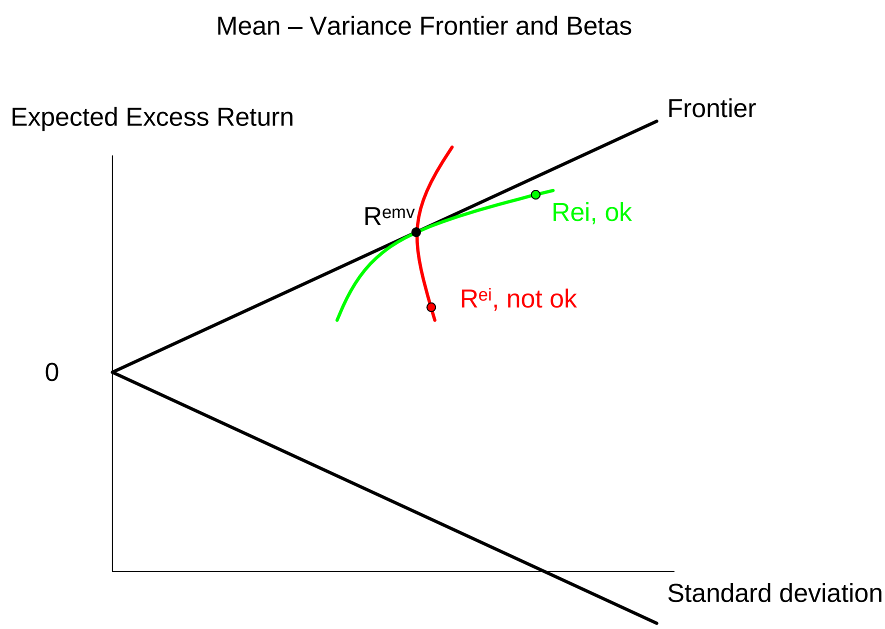
(g) *Conclusion:* For traders: here is the SR you can get from investing. If the boss says "market neutral" then the second term is the max SR you can get from your alpha machine.

(h) *Conclusion:* For economists. When the traders are done, $\alpha' cov(\varepsilon,\varepsilon')^{-1}\alpha$ should be reasonable. If $cov(\varepsilon,\varepsilon')$ is small, so should alpha.

9. A direct proof that mean-variance efficiency implies a single-factor model, 35000 style. reminder. Suppose R^{emv} is on the mean-variance frontier, meaning it has maximum Sharpe ratio. Suppose you form a portfolio that shades a bit in the direction of a particular security[8], i.e. $R^{ep} = R^{emv} + \varepsilon R^{ei}$. If R^{emv} is on the mean-variance frontier, then this move must have the same Sharpe ratio as the R^{emv} portfolio (the green R^{ei}, ok case) If it increased the Sharpe

[8] Portfolio weights don't have to add to one here, since these are excess returns.

ratio, then, the original portfolio was not on the mvf. If it decreased the Sharpe ratio, then going in the other direction, shorting R^e would increase the Sharpe ratio (the Rei, not ok case). See the drawing.



Let's figure out the change in mean and standard deviation of your portfolio from adding a very small ε

$$\begin{aligned} E\left(R^{ep}\right) &= E\left(R^{emv}\right) + \varepsilon E\left(R^{ei}\right) \\ \frac{dE\left(R^{ep}\right)}{d\varepsilon} &= E\left(R^{ei}\right) \end{aligned}$$

$$\begin{aligned} \sigma(R^{ep}) &= \sqrt{\sigma^2\left(R^{emv}\right) + \varepsilon^2\sigma^2\left(R^{ei}\right) + 2\varepsilon cov(R^{emv}, R^{ei})} \\ \frac{d\sigma(R^{ep})}{d\varepsilon} &= \frac{1}{2}\left(\cdot\right)^{-\frac{1}{2}} \times \left[2\varepsilon\sigma^2\left(R^{ei}\right) + 2cov(R^{emv}, R^{ei})\right] \\ \left.\frac{d\sigma(R^{ep})}{d\varepsilon}\right|_{\varepsilon=0} &= \frac{cov(R^{emv}, R^{ei})}{\sigma(R^{emv})} = \beta_{i,R^{emv}}\sigma(R^{emv}) \end{aligned}$$

(Words: If you add a small amount of R^{ei} to the portfolio R^{emv}, the volatility of your *portfolio* goes up by $\beta_{i,R^{emv}}\sigma(R^{emv})$. $\sigma(R^{ei})$ does not matter!)

(a) Now, if we're going to have the "ok" case from the drawing, it must be that

$$\begin{aligned} \frac{dE\left(R^{ep}\right)}{d\varepsilon} &= \frac{E(R^{emv})}{\sigma(R^{emv})} \left.\frac{d\sigma(R^{ep})}{d\varepsilon}\right|_{\varepsilon=0} \\ E\left(R^{ei}\right) &= \frac{E(R^{emv})}{\sigma(R^{emv})}\beta_{i,R^{emv}}\sigma(R^{emv}) \\ E\left(R^{ei}\right) &= \beta_{i,R^{emv}}E(R^{emv}) \end{aligned}$$