26 Utility functions

26.1 Utility function algebra

Habits

- $$\Delta c_{t+1} = g + v_{t+1}$$

$$U_t = E\sum \beta^t \frac{1}{1-\gamma}\left(C_t - X_t\right)^{1-\gamma}$$

•"external habit",

$$\frac{\partial U}{\partial C_t} = \left(C_t - X_t\right)^{-\gamma}$$

$$\begin{aligned}
u_{cc} &= -\gamma\left(C_t - X_t\right)^{-\gamma-1} \\
-\frac{Cu_{cc}}{u_c} &= \frac{-\gamma C\left(C_t - X_t\right)^{-\gamma-1}}{\left(C_t - X_t\right)^{-\gamma}} = \frac{-\gamma C}{C-X} = \frac{-\gamma}{\frac{C-X_t}{C}} = \frac{-\gamma}{S_t} \\
S_t &= \frac{C-X}{C} = \text{ state variable.}
\end{aligned}$$

$$m_{t+1} = \beta\frac{\left(C_{t+1} - X_{t+1}\right)^{-\gamma}}{\left(C_t - X_t\right)^{-\gamma}} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{\left(1-\frac{X_{t+1}}{C_{t+1}}\right)^{-\gamma}}{\left(1-\frac{X_t}{C_t}\right)^{-\gamma}} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}$$

•Internal?

$$\frac{\partial U_t}{\partial c_t} = \left(C_t - X_t\right)^{-\gamma} - E_t\sum_{j=0}^{\infty}\beta^j\left(C_{t+j} - X_{t+j}\right)^{-\gamma}\frac{\partial X_{t+j}}{\partial C_t}$$

Fact: external vs. internal makes little difference. Exact: power utility, AR(1) habit, const Rf, then $MU_t = k \times \left(C_t - X_t\right)^{-\gamma}$. (CC appendix)

Preview: dS_{t+1}/dC_{t+1} is big, so $\sigma(m)$ is big. S_{t+1} is heterosckeadstic, hence $\sigma_t(m)$ varies

•Slow-moving habit.

$$\begin{aligned}
X_t &= \sum \phi^j C_{t-j} \\
X_t &= \phi X_{t-1} + C_t \\
\frac{X_t}{C_t} &= \phi\frac{X_{t-1}}{C_{t-1}}\left(\frac{C_{t-1}}{C_t}\right) + 1
\end{aligned}$$

Instead, AR(1) for log S

$$\Delta s_{t+1} = -(1-\phi)(s_t - \bar{s}) + \lambda(s_t)(\Delta c_{t+1} - g)$$

•

$$\begin{aligned} \ln m_{t+1} &= \ln \beta - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} \\ &= -\delta - \gamma \Delta c_{t+1} - \gamma \left[-(1-\phi)(s_t - \bar{s}) + \lambda(s_t)(\Delta c_{t+1} - g) \right] \\ \ln m_{t+1} &= -\delta - \gamma g - \gamma \left[1 + \lambda(s_t) \right] (\Delta c_{t+1} - g) + \gamma (1-\phi)(s_t - \bar{s}) \end{aligned}$$

Note the same structure as we have seen in term structure models.

• Rf

$$\begin{aligned} r^f &= -\ln E_t m_{t+1} \\ &= \delta + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma^2 \sigma^2}{2} \left[1 + \lambda(s_t) \right]^2 \end{aligned}$$

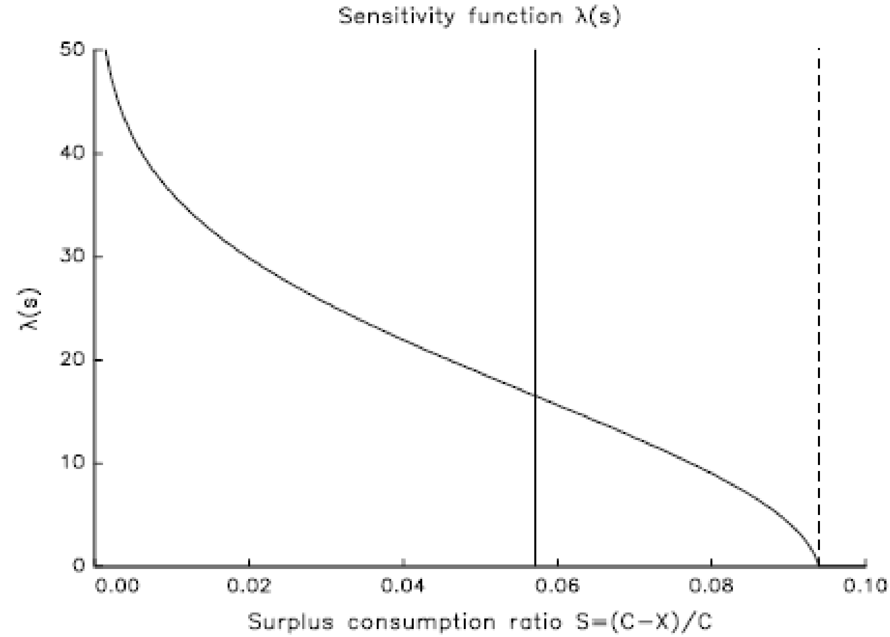
Our form:

$$\begin{aligned} 1 + \lambda(s_t) &= \frac{1}{\bar{S}} \sqrt{1 - 2(s_t - \bar{s})} \\ \bar{S} = e^{\bar{s}} &= \sqrt{\frac{\gamma \sigma^2}{1-\phi}} \end{aligned}$$

•R^f result: constant

$$\begin{aligned} r^f &= -\ln \beta + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{\gamma}{2}(1-\phi)\left[1 - 2(s_t - \bar{s}) \right] \\ &= -\ln \beta + \gamma g - \frac{\gamma}{2}(1-\phi) \end{aligned}$$

•Plot: a square root function of log s, meaning



•Back to

$$\ln m_{t+1} = -\gamma g - \gamma \left[1 + \lambda\left(s_t\right)\right]\left(\Delta c_{t+1} - g\right) + \gamma\left(1 - \phi\right)\left(s_t - \bar{s}\right)$$

a) $\lambda \approx 20$ so big amplification

b) A conditionally heteroskedastic m! Just what we need to generate time varying risk premia! (See CP, bonds)

c) A scaled factor model/conditional consumption based model. $a - b(s_t)\Delta c_{t+1}$

• Main results:

$$\frac{P_t}{D_t}(S_t) = E_t\left(M_{t+1}\left(1 + \frac{P_{t+1}}{D_{t+1}}(S_{t+1})\right)\frac{D_{t+1}}{D_t}\right)$$

See tables p. 470. Parameters $\gamma = 2$,..

• Long run Equity premium:

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}$$

Short run: dS/dC is large, big $\sigma(m)$

Long run? S_t is stationary, and over long run becomes uncoupled with C. $\sigma(m) \to \gamma\sigma(\Delta c) - \gamma\sigma(\Delta s) \to \gamma\sigma(\Delta c)$

Answer: $S^{-\gamma}$ variance explodes – S has a "fat left tail"

Summary:

- Equity premium and constant, low Rf.
- Time-varying risk aversion, time-varying ER, at root of many puzzles – main point.
- Risk aversion *is* high – as in every other model so far.
- "Precautionary saving" solves volatile Rf of habit models
- Long run equity premium – Most models with stationary S can't do it.

$$m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{S_{t+1}}{S_t} \right)^{-\gamma}$$

• Varying, high risk premium, constant risk free rate: habits and *temporal* nonseparabilities also "separate intertemporal substitution and risk aversion" – don't need recursive utility for this purpose.

Nonseparable across states – Epstein Zin, recurisive utility

$$U_t = \left((1-\beta)c_t^{1-\rho} + \beta \left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}.$$

γ = risk aversion $\rho = 1/\text{eis}$. power utility for $\rho = \gamma$.

Major results

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} \left(\frac{U_{t+1}}{\left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma}.$$

Again, standard form. Get U to have needed properties?

1. Using R^c= claim to consumption to proxy for $E_t U_{t+1}$

$$m_{t+1} = \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} \right]^{\theta} \left(\frac{1}{R_{t+1}^c} \right)^{1-\theta},$$

$$\theta = \frac{1-\gamma}{1-\rho}.$$

2. U from news of future consumption! ($\rho \approx 1$).

$$(E_{t+1} - E_t) \ln m_{t+1} \approx -\gamma (E_{t+1} - E_t)(\Delta c_{t+1}) + (1-\gamma) \left[\sum_{j=1}^{\infty} \beta^j (E_{t+1} - E_t)(\Delta c_{t+1j}) \right]$$

News about *future* long-horizon consumption growth enters the *current* period m.

Note: unlike habits, $\sigma_t(m_{t+1})$ must come from σ_t of long run consumption process. Thus, paired with VAR models that imply big variation in the right hand term.

Bansal Yaron Kiku

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho x_t + \phi_e \sigma_t e_{t+1} \\
\sigma^2_{t+1} &= \bar{\sigma}^2 + v(\sigma^2_t - \bar{\sigma}^2) + \sigma_w w_{t+1} \\
\Delta d_{t+1} &= \mu_d + \phi x_t + \pi \sigma_t \eta_{t+1} + \phi \sigma_t u_{d,t+1}
\end{aligned}$$

Algebra

1.
$$\left.\frac{\partial}{\partial \xi} U_t(c_t - p_t\xi, c_{t+1} + x_{t+1}\xi)\right|_{\xi=0} = 0.$$

$$U_t = \left((1-\beta) c_t^{1-\rho} + \beta \left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$

$$\frac{\partial U_t}{\partial c_t} = \frac{1}{1-\rho} U_t^{\rho} (1-\beta) c_t^{-\rho}.$$

Then it's just a massive application of the chain rule.

$$\frac{\partial U_t}{\partial c_t} p_t = \frac{1}{1-\rho} U_t^{\rho} \beta \frac{1-\rho}{1-\gamma} \left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{\gamma-\rho}{1-\gamma}} \left[E_t \left((1-\gamma) U_{t+1}^{-\gamma} \frac{\partial U_{t+1}}{\partial c_{t+1}} x_{t+1} \right) \right]$$

$$p_t (1-\beta) c_t^{-\rho} = \beta \frac{1-\rho}{1-\gamma} \left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{\gamma-\rho}{1-\gamma}} \left[E_t \left((1-\gamma) U_{t+1}^{-\gamma} \frac{1}{1-\rho} U_{t+1}^{\rho} (1-\beta) c_{t+1}^{-\rho} x_{t+1} \right) \right]$$

$$p_t c_t^{-\rho} = \beta \left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{\gamma-\rho}{1-\gamma}} \left[E_t \left(U_{t+1}^{\rho-\gamma} c_{t+1}^{-\rho} x_{t+1} \right) \right]$$

Thus, defining the discount factor from $p_t = E(m_{t+1} x_{t+1})$,

$$m_{t+1} = \beta \left(\frac{U_{t+1}}{\left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{c_{t+1}}{c_t} \right)^{-\rho}$$

2. *market return* The basic idea – exploit linear homogeneity.

$$U_t = E_t \sum_{j=0}^{\infty} \frac{\partial U_t}{\partial c_{t+j}} c_{t+j}$$

$$\frac{U_t}{\partial U_t / \partial c_t} = E_t \sum_{j=0}^{\infty} m_{t,t+j} c_{t+j} = W_t$$

Then

$$R^c_{t+1} = \frac{W_{t+1} + c_{t+1}}{W_t}$$

Note 1) It must be all wealth portfolio, claim to all consumption. 2) It must be all consumption, not just nondurable and services

Constantinides and Duffie – idiosyncratic risk

$$m = \beta \left(e^{\frac{\gamma(\gamma+1)}{2} y^2_{t+1}} \right) \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

y_{t+1} =*cross-sectional variance* of consumption growth. y_{t+1} is known at time t+1,

$$\Delta c^i_{t+1} = \Delta c_{t+1} + \eta_{i,t+1} y_{t+1} - \frac{1}{2} y^2_{t+1};\ \sigma^2 \left(\eta_{i,t+1} \right) = 1$$

a) check

$$E \left(\frac{C^i_{t+1}}{C^i_t} | \Delta C_{t+1} \right) = e^{\Delta c^i_{t+1} - \frac{1}{2} y^2_{t+1} + \frac{1}{2} y^2_{t+1}} = \frac{C_{t+1}}{C_t}$$

yes, it's an "idiosyncratic shock." Note permanent – keeps people from saving up.

b) derivation Now an exponential version of the projection argument, where $1/2\sigma^2$ terms do the pricing.

$$\begin{aligned}
1 &= E \left[\beta \left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\gamma} R_{t+1} \right] \\
&= E \left[\beta E \left[\left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\gamma} | agg_{t+1} \right] R_{t+1} \right] \\
&= E \left[\beta E \left[e^{-\gamma \left(\Delta c_{t+1} + \eta_{i,t+1} y_{t+1} - \frac{1}{2} y^2_{t+1} \right)} | agg_{t+1} \right] R_{t+1} \right] \\
&= E \left[\beta e^{-\gamma \Delta c_{t+1} + \gamma \frac{1}{2} y^2_{t+1} + \frac{1}{2} \gamma^2 y^2_{t+1}} R_{t+1} \right] \\
&= E \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} e^{\frac{1}{2} \gamma(\gamma+1) y^2_{t+1}} R_{t+1} \right]
\end{aligned}$$

• Absolutely brilliant existence / reverse engineering theorem! Pick y_{t+1} to get anything! Literature "calibrated" got nowhere (typical, saved up and avoided)

• Quantitatively true? *is* y_{t+1} what we need? (Remember *consumption)*

$$\sigma(m) = \sigma \left(e^{\frac{1}{2} \gamma(\gamma+1) y^2_{t+1}} \right) \approx \sigma \left(\frac{1}{2} \gamma(\gamma+1) y^2_{t+1} \right)$$

$\gamma = 1$ $\sigma(y^2_{t+1}) = 0.5$. $y_{t+1} = \sigma(\Delta c_{it+1}) = 0.71$???. But this is the *variation,* not the *level.* in some years more, in some year less.

•Soo...needs huge γ (just like habits). Solve Rf with huge γ? is $\sigma_t(m)$ correlated with $E_t(r)$? (what moment of Δc_i??) don't know yet.

Garleanu/Panageas heterogenous risk aversion (complete markets!)

1.
$$\max E\int e^{-\delta t}\frac{c_{At}^{1-\gamma_A}}{1-\gamma_A}dt+\lambda\int e^{-\delta t}\frac{c_{Bt}^{1-\gamma_B}}{1-\gamma_B}\quad s.t.\quad c_{At}+c_{Bt}=c_t$$
$$c_{At}^{-\gamma_A}=\lambda c_{Bt}^{-\gamma_B}$$

Sharing rule result:

$$\lambda^{\frac{1}{\gamma_B}}c_{At}^{\frac{\gamma_A}{\gamma_B}}+c_{At}=c_t$$
$$\lambda^{-\frac{1}{\gamma_A}}c_{Bt}^{\frac{\gamma_B}{\gamma_A}}+c_{Bt}=c_t$$

Proof:

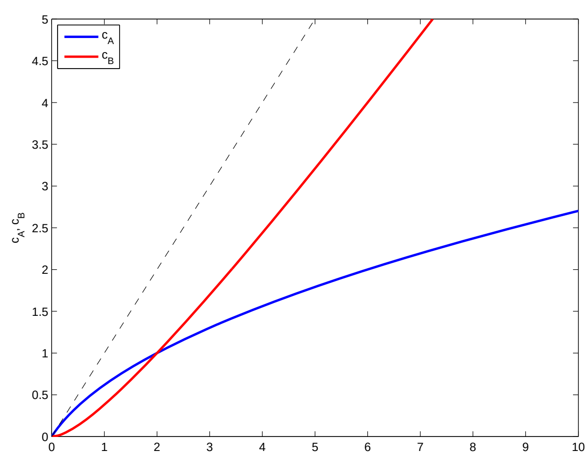
$$c_{At}=\lambda^{-\frac{1}{\gamma_A}}c_{Bt}^{\frac{\gamma_B}{\gamma_A}}$$
$$\lambda^{-\frac{1}{\gamma_A}}c_{Bt}^{\frac{\gamma_B}{\gamma_A}}+c_{Bt}=c_t$$

similarly

$$c_{Bt}=\lambda^{\frac{1}{\gamma_B}}c_{At}^{\frac{\gamma_A}{\gamma_B}}$$
$$\lambda^{\frac{1}{\gamma_B}}c_{At}^{\frac{\gamma_A}{\gamma_B}}+c_{At}=c_t$$

2. Sketch: For $\gamma_A/\gamma_B=2$,

$$c_{Bt}^{\frac{1}{2}}+c_{Bt}=c_t$$
$$c_{At}^{2}+c_{At}=c_t$$



3. Risk premiums:

$$\frac{dc}{c} = \mu dt + \sigma dz$$

$$\frac{E_t\left(dR\right) - rdt}{\sigma_t(dR)} \leq \gamma_A \sigma_t \left(\frac{dc_A}{c_A}\right) = \left(\frac{1}{\gamma_B}\frac{c_{Bt}}{c_t} + \frac{1}{\gamma_A}\frac{c_{At}}{c_t}\right)^{-1}\sigma$$

Proof:

$$\lambda^{\frac{1}{\gamma_B}} c_{At}^{\frac{\gamma_A}{\gamma_B}} + c_{At} = c_t$$

$$\left[\lambda^{\frac{1}{\gamma_B}}\frac{\gamma_A}{\gamma_B}c_{At}^{\frac{\gamma_A}{\gamma_B}-1} + 1\right]dc_A + \left\{\frac{1}{2}\lambda^{\frac{1}{\gamma_B}}\left(\frac{\gamma_A}{\gamma_B}\right)\left(\frac{\gamma_A}{\gamma_B}-1\right)c_{At}^{\frac{\gamma_A}{\gamma_B}-2}dc_{At}^2\right\} = dc_t$$

$$\left[\lambda^{\frac{1}{\gamma_B}}\frac{\gamma_A}{\gamma_B}\frac{c_{At}^{\frac{\gamma_A}{\gamma_B}}}{c} + \frac{c_A}{c}\right]\frac{dc_A}{c_A} + \left\{\frac{1}{2}\lambda^{\frac{1}{\gamma_B}}\left(\frac{\gamma_A}{\gamma_B}\right)\left(\frac{\gamma_A}{\gamma_B}-1\right)\frac{c_{At}^{\frac{\gamma_A}{\gamma_B}}}{c}\frac{dc_{At}^2}{c_A^2}\right\} = \frac{dc_t}{c_t}$$

$$\left[\lambda^{\frac{1}{\gamma_B}}\frac{\gamma_A}{\gamma_B}\frac{c_{At}^{\frac{\gamma_A}{\gamma_B}}}{c} + \frac{c_A}{c}\right]\sigma_A = \sigma$$

Using

$$c_B = \lambda^{\frac{1}{\gamma_B}} c_A^{\frac{\gamma_A}{\gamma_B}}$$

we then have

$$\left[\frac{\gamma_A}{\gamma_B}\frac{c_{Bt}}{c} + \frac{c_A}{c}\right]\sigma_A = \sigma$$

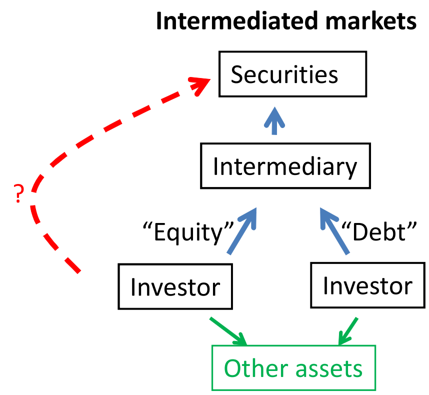
$$\sigma_A = \frac{\sigma}{\frac{\gamma_A}{\gamma_B}\left(\frac{c_{Bt}}{c_t}\right) + \frac{c_{At}}{c_t}} = \frac{1}{\gamma_A}\frac{\sigma}{\frac{1}{\gamma_B}\frac{c_{Bt}}{c_t} + \frac{1}{\gamma_A}\frac{c_{At}}{c_t}}$$

More

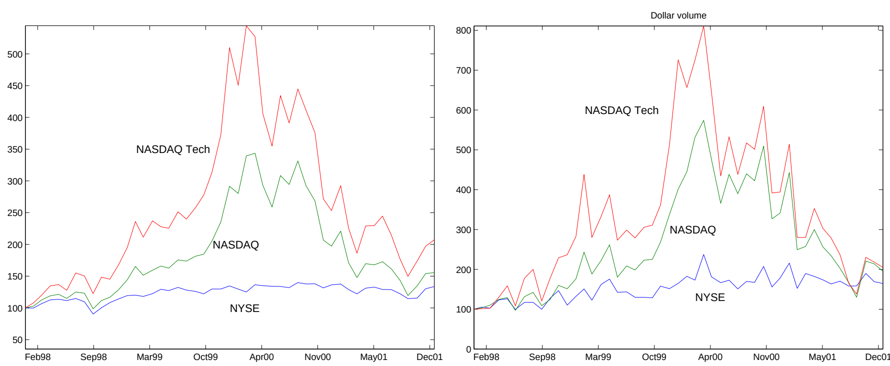
- Hansen "distorted beliefs"

$$1 = E\left(\frac{M_{t+1}}{M_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{t+1}\right)$$

- Models with friction, leverage, etc., where sdf is disconnected from the representative agent. Warning "marginal buyer" fallacy



- Does trading matter for prices? Every "bubble" has been a "trading frenzy" ("Stocks as money")



Production side and general equilibrium

- Q theory. Without adjustment costs, Q=1, investors can be as silly as they want, *supply* constrains risk premiums.
- From Problem set 2:

$$\pi_t = \theta_t k_t - \left[1 + \frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)\right] i_t$$

$$V_t(k_t, \cdot) = \max_{\{i_t\}} E_t \sum_{j=0}^{\infty} m_{t+j}\pi_{t+j}$$

$$s.t.\ k_{t+1} = (1-\delta)(k_t + i_t)$$

$$1 + \alpha \frac{i_t}{k_t} = Q_t = \frac{W_t}{(k_t + i_t)}$$

$$\frac{i_t}{k_t} = \frac{1}{\alpha}\left[Q_t - 1\right].$$

$$R^I_{t+1} = (1-\delta)\frac{1+\theta_{t+1}+\alpha\left(\frac{i_{t+1}}{k_{t+1}}\right)+\frac{\alpha}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2}{1+\alpha\left(\frac{i_t}{k_t}\right)}$$

$$dR^I_t = \frac{\left[\theta_t - \delta - \frac{\alpha}{2}\left(\frac{i_t}{k_t}\right)^2\right]dt + \alpha\left(\frac{i_t}{k_t}\right)\frac{di_t}{i_t}}{1+\alpha\left(\frac{i_t}{k_t}\right)}$$

$$R_{t+1} = R^I_{t+1}$$

- From "Production based asset pricing"

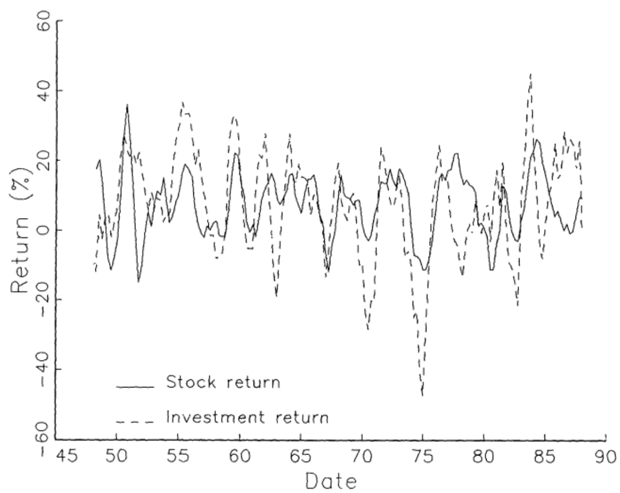


Figure 2. Quarterly observations of annual (from t − 4 to t) real returns on the value weighted NYSE portfolio, and annual investment returns.

$$E_t R^I_{t+1} = E_t R_{t+1}$$

Table III

Forecasts of Stock Returns and Investment Returns

The forecasting variables are as follows: Term is the 10-year government bond return less treasury bill return. Corp is the corporate bond return less the treasury bill return. Ret is the real value weighted return. d/p is the dividend-price ratio. I/k is the investment/capital ratio. Term and d/p are based on returns for the year ending in the indicated quarter (t − 5 or t − 2), Ret and Corp are returns for the quarter t − 5 or t − 2. The data sample is 1947:1–1987:4.

"β" gives OLS slope coefficients. "% p value" gives the percent probability values of two sided t-tests of the corresponding slope coefficients. "Joint χ^2" gives the percent probability values for a χ^2 test of the joint significance of the coefficients. "Joint χ^2 all but d/p" gives the percent probability value of a χ^2 test for the joint significance of all variables except the dividend-price ratio. "Regressions without d/p" give partial results for corresponding multiple regressions that exclude the dividend-price ratio.

Annual return standard errors are adjusted using a Hansen (1982)–Newey-West (1987) correction, using 8 covariances, or twice the overlap. All correlation standard errors include this correction.

Panel A. Single Regression

1. Quarterly Returns

$$\text{Return}(t-1 \rightarrow t) = \alpha + \beta X(t-2) + \varepsilon(t)$$

Forecasting Variable	Stock Return		Investment Return		Stock—Inv.
	β	% p value	β	% p value	% p value
Term	0.16	0.53	0.10	0.05	24.10
Corp	0.35	0.94	0.16	0.23	12.44
Ret	0.16	2.51	0.15	0.00	88.56
d/p	1.32	0.26	0.11	70.70	1.22
I/k	−1.53	2.12	−1.71	0.00	79.96

2. Annual Returns

$$\text{Return}(t-4 \text{ to } t) = \alpha + \beta X(t-5) + \varepsilon(t)$$

Forecasting Variable	Stock Return		Investment Return		Stock—Inv.
	β	% p value	β	% p value	% p value
Term	0.35	1.12	0.35	2.51	99.57
Corp	0.68	1.23	0.59	0.32	70.99
Ret	0.12	50.97	0.24	0.66	48.86
d/p	5.02	0.28	0.80	48.47	0.02
I/k	−4.74	4.34	−7.40	0.00	25.35

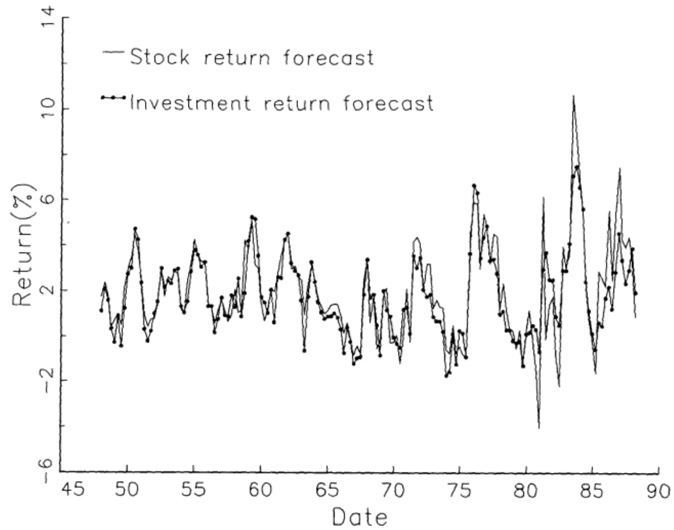


Figure 3. Forecasts of quarterly stock returns and investment returns. Forecasts are from linear regressions of returns on the term premium, corporate premium, lagged return and investment to capital ratio.

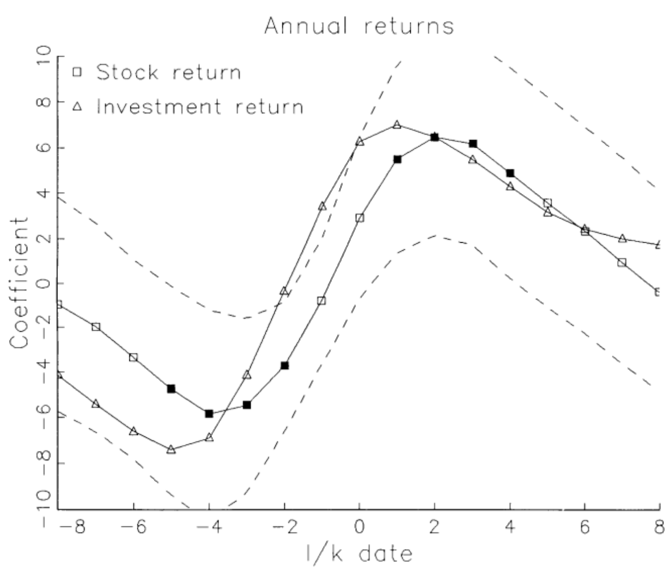
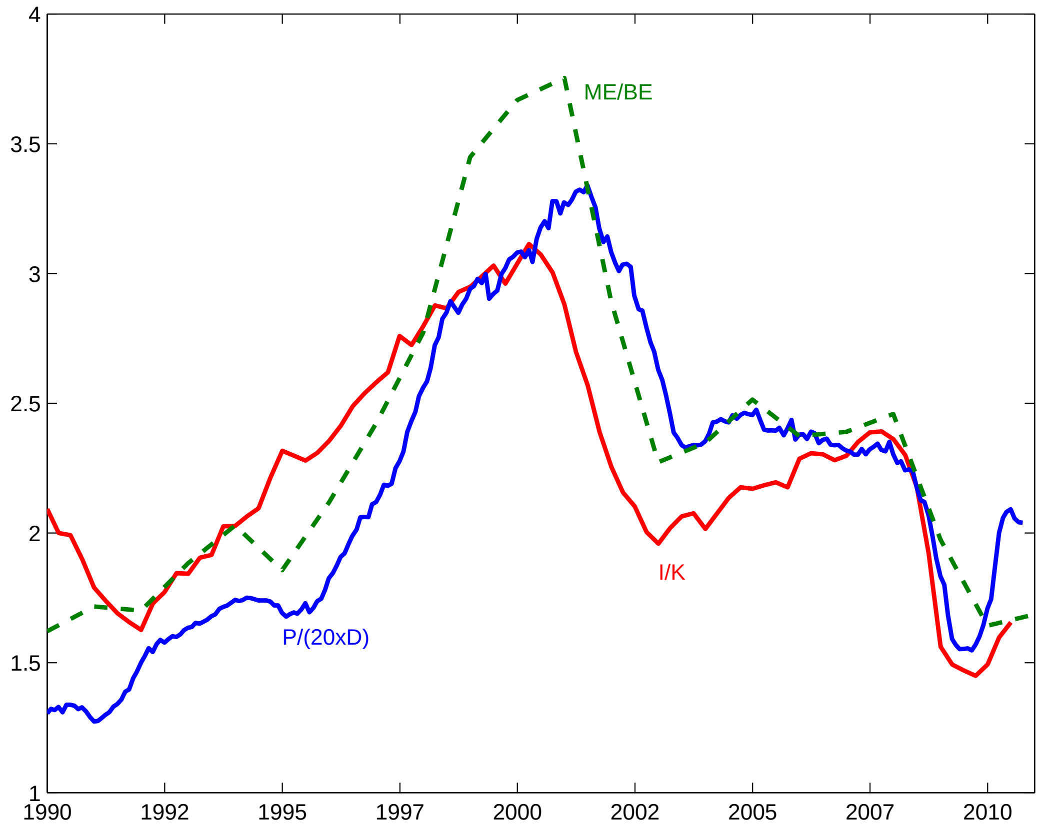


Figure 4. Single regression slope coefficients of quarterly (from t − 1 to t) and annual (from t − 4 to t) investment returns and stock returns on investment/capital ratios. The

- From "Discount rates"



$$1+\alpha\frac{i_t}{k_t}=\frac{\text{market}_t}{\text{book}_t}=Q_t$$

- Moral: Just because they say "Q theory doesn't work" don't believe them!
- Challenge: technologies that allow producers to transfer output *across states of nature?* Two-field example.

Two Trees

- Rebalance conundrum

$$U=E\int e^{-\delta\tau}\ln C_{t+\tau}d\tau$$

$$\frac{dD_i}{D_i}=\mu dt+\sigma dZ_i$$

$$s=\frac{D_1}{D_1+D_2};ds=...$$

$$\frac{P}{C} = E_t \int e^{-\delta\tau} \frac{1}{C_{t+\tau}} D_{t+\tau} d\tau = E_t \int e^{-\delta\tau} s_{t+\tau} d\tau$$

($\delta = \sigma^2$ simple case)

$$\frac{P}{C} = \frac{1}{2\delta} \left[1 + \left(\frac{1-s}{s} \right) \ln(1-s) - \frac{s}{1-s} \ln(s) \right]$$

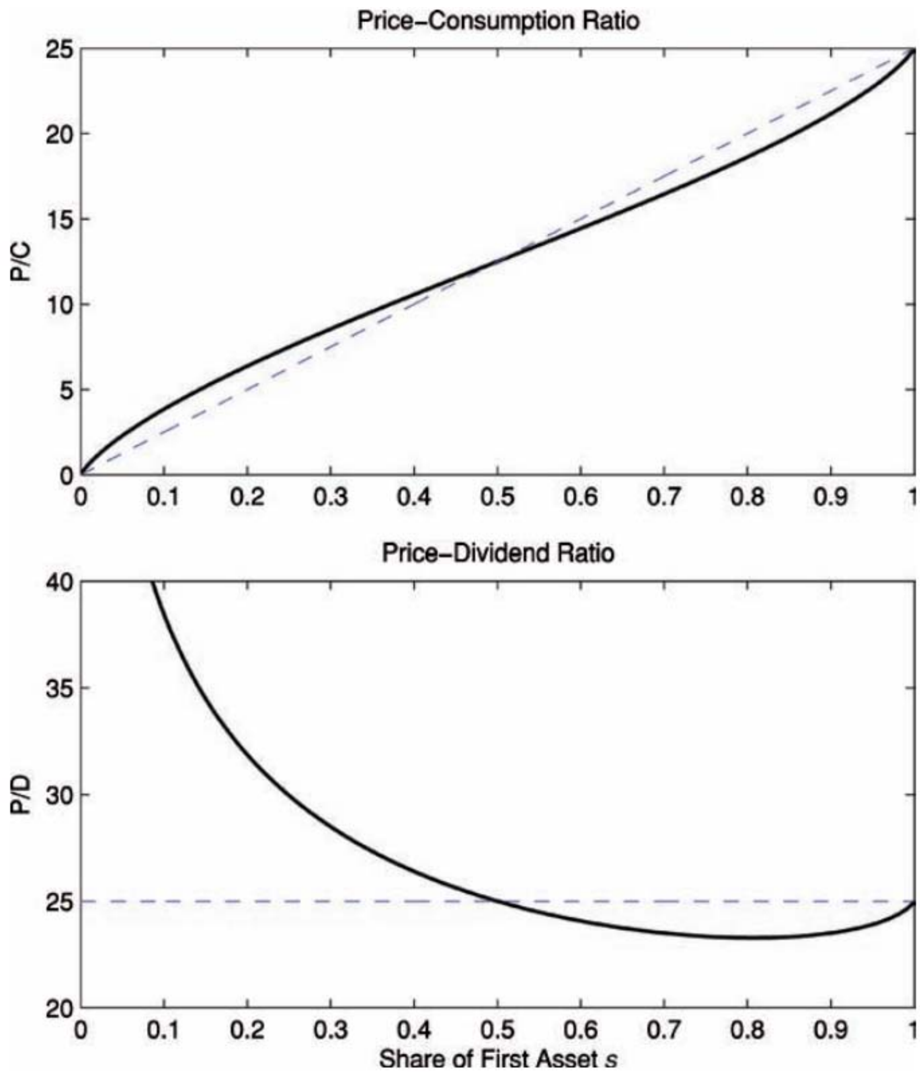


Figure 1
Price-consumption ratio and price-dividend ratio
The top panel presents the price-consumption ratio of the first asset, in the simple case, using parameters $\mu = 0.02$, $\delta = \sigma^2 = 0.04$. The bottom panel presents the price-dividend ratio. The dashed line in the top panel gives the 45-degree line; the dashed line in the bottom panel gives the price-dividend ratio of the market portfolio and of the one-tree model, $1/\delta = 25$.

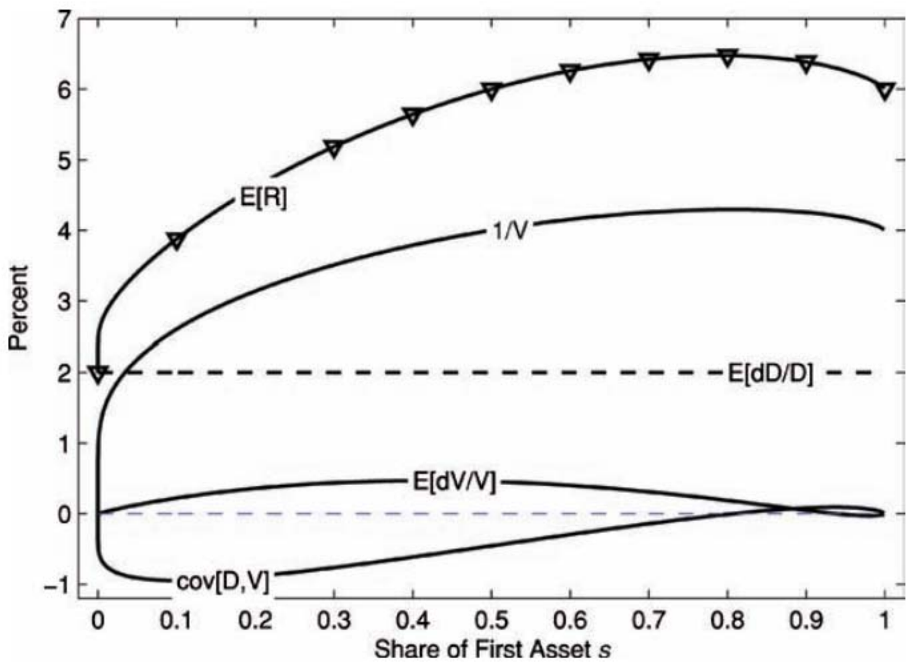


Figure 2
Expected return and components
$E[R]$ gives the expected return of the first asset as a function of the share of the first asset. The remaining lines give components of this expected return. $1/V$ gives the dividend-price ratio. $E[dD/D]$ gives the expected dividend-growth rate. $E[dV/V]$ gives the expected change in the price-dividend ratio. Cov$[D,V]$ gives Cov$[dV/V,dD/D]$, the covariance of dividend growth with price-dividend ratio shocks.

- "Under-reaction" "momentum" in small stocks, "over reaction" mean reversion in big stocks

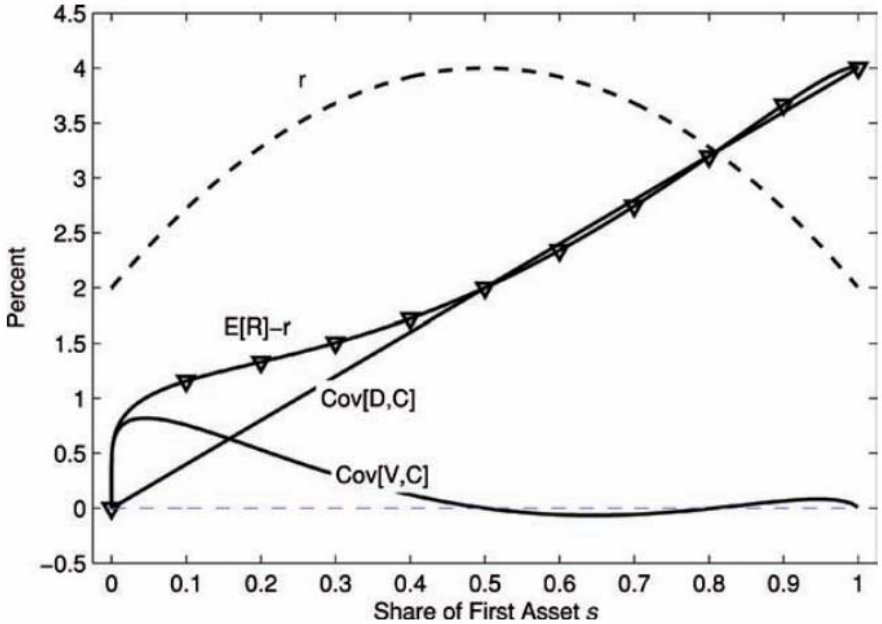


Figure 3
Expected excess return and components
$E[R] - r$ gives the expected excess return of the first asset as a function of its share. $\mathrm{Cov}[D,C]$ gives the covariance of dividend and consumption-growth shocks, $\mathrm{Cov}[\mathrm{d}D/D,\mathrm{d}C/C]$. $\mathrm{Cov}[V,C]$ gives the covariance of price-dividend ratio and consumption-growth shocks, $\mathrm{Cov}[\mathrm{d}V/V,\mathrm{d}C/C]$. The latter two components add up to the expected excess return. The riskless rate is given by r.

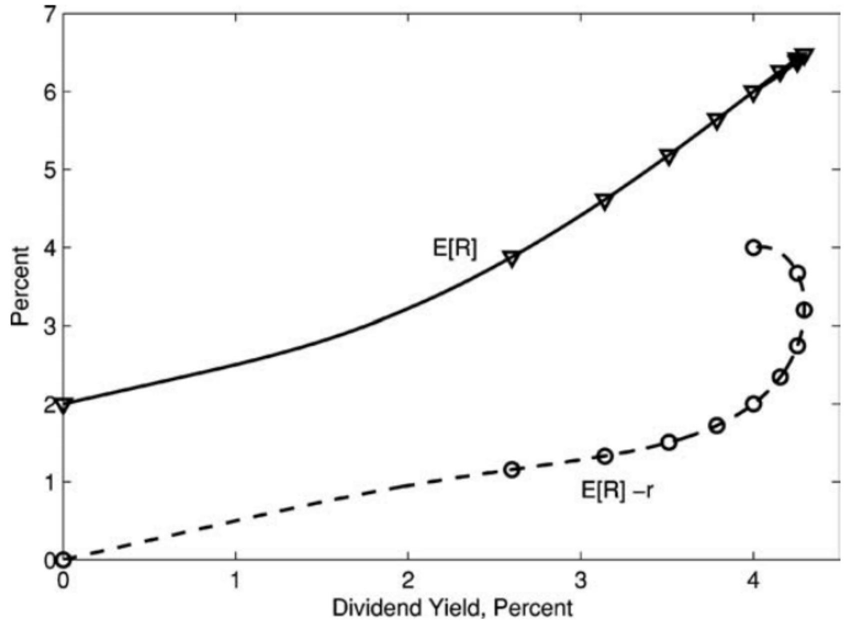


Figure 5
Dividend yields and expected returns
The solid line plots the first asset's expected return versus its dividend yield. The dashed line plots the first asset's expected excess return versus its dividend yield. Symbols mark the points $s = 0$, $s = 0.1$, $s = 0.2$, etc., starting from the left.

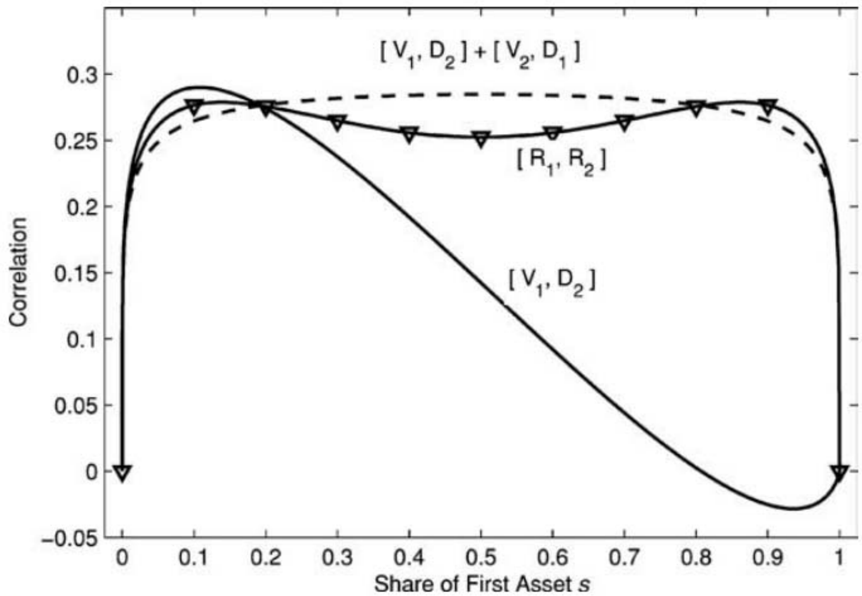


Figure 7
Return correlation
The solid line with triangles labeled $[R_1, R_2]$ gives the conditional correlation between the two assets' returns, given the dividend share s of the first asset. The remaining lines give components of that correlation, using the decomposition of Equation (40). For example, $[V_1, D_2]$ gives the component of correlation corresponding to $\text{Cov}\left[\frac{dV_1}{V_1}, \frac{dD_2}{D_2}\right]$.

- "Contagion" D1 rises, ER2 declines, P2 rises.

Morals:

1) Be very careful about numerical solutions! (Boundaries here cause a lot of problems)

2) The traditional case assumes linear technology = no adjustment costs. This case = endowment economy, infinite adjustment costs. Agenda: Finite adjustment costs, short run "two tree" dynamics, long-run rebalancing?

Risk Sharing is better than you think

- Point

$$\ln \frac{e_{t+1}}{e_t} = \ln m_{t+1}^f - \ln m_{t+1}^d$$

$$\sigma^2 \left(\ln \frac{e_{t+1}}{e_t} \right) = \sigma^2 \left(\ln m_{t+1}^f \right) + \sigma^2 \left(\ln m_{t+1}^d \right) - 2cov(\ln m_{t+1}^f, \ln m_{t+1}^d)$$

$$15\%^2 = 40\%^2 + 40\%^2 + ???$$

- Survives incomplete markets – true of x^* so $\sigma(m)$ even bigger

- One good + transport costs vs. two goods (tradeable + nontradeable) and limited substitution international economics.

Risk sharing requires frictionless goods markets. The container ship is a risk sharing innovation as important as 24 hour trading. Suppose that Earth trades assets with Mars by radio, in complete and frictionless capital markets. If Mars enjoys a positive shock, Earth-based owners of Martian assets rejoice in anticipation of their payoffs. But trade with Mars is still impossible, so the real exchange rate between Mars and Earth must adjust exactly to offset any net payoff. In the end, Earth marginal utility growth must reflect Earth resources, and the same for Mars. Risk sharing is impossible. If the underlying shocks are uncorrelated, the exchange rate variance is the sum of the variances of Earth and Mars marginal utility growth, and we measure a zero risk sharing index despite perfect capital markets.

At the other extreme, if there is costless trade between the two plantes, and the real exchange rate is therefore constant, marginal utilities can move in lockstep.