Week 2 Summary:

Value = value to investor, willingness to pay for a little more x_{t+1}

$p_t = E_t\left[m_{t+1}x_{t+1}\right];\quad m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \approx 1-\delta-\gamma\Delta c_{t+1}$

Equivalent representations and implications:

1. $p_t = E_t\sum_{j=1}^{\infty} m_{t,t+j}d_{t+j};\quad p_t = E_t\int_{s=0}^{\infty}\frac{\Lambda_{t+s}}{\Lambda_t}D_{t+s}ds$
2. $1 = E_t(m_{t+1}R_{t+1})\ 0 = E(m_{t+1}R^e_{t+1});$
3. $0 = E_t\left[d\left(\Lambda_t V_t\right)\right] = E_t\left[d\left(\Lambda_t p_t\right)\right] + \Lambda_t D_t dt\ ;\ \ 0 = E_t\left[\frac{d(\Lambda_t V_t)}{\Lambda_t V_t}\right] = E_t\left[\frac{d(\Lambda_t p_t)}{\Lambda_t p_t}\right] + \frac{D_t}{P_t}dt$
4. $\Lambda_t = e^{-\delta t}u'(c_t) = e^{-\delta t}c_t^{-\gamma};$
$$\begin{aligned}\frac{d\Lambda_t}{\Lambda_t} &= -\delta dt - \gamma\frac{dc_t}{c_t} + \frac{1}{2}\gamma(\gamma+1)\frac{dc_t^2}{c_t^2}\\ \frac{d\Lambda_t}{\Lambda_t} &= -\delta dt - \gamma d\log c_t + \frac{1}{2}\gamma^2 d\left(\log c_t\right)^2\end{aligned}$$
5. $R^f = 1/E(m_{t+1})\qquad r_t^f dt = -E_t\left[\frac{d\Lambda_t}{\Lambda}\right]\quad r_t^f = \delta + \gamma E_t\left(d\log c_t\right) - \frac{1}{2}\gamma^2\sigma_t^2\left(d\log c_t\right)$
(Prediction for r^f, intertemporal substitution and precautionary savings motives in interest rates)
6. $p = \frac{E(x)}{R^f} + cov(m,x)$
(Cost of capital, risk discount covariances. A lot prettier than $p^i = \frac{E(x^i)}{E(R^i)}$)
7. $E(R^{ei}) = R^f cov(m, R^{ei}) = \beta_{i,m}\lambda_m;\quad E_t(dR_t^i) - r_t^f dt = E_t\left(\frac{d\Lambda_t}{\Lambda_t}dR_t^i\right) = \beta_{i,\Lambda,t}\lambda_{\Lambda,t}$
8. $E_t\left(dR_t^i\right) - r^f dt = \gamma cov_t\left(dR_t^i, \frac{dc_t}{c_t}\right) = \beta_{i,\frac{dc}{c}}\lambda_c,\ \ \lambda_c = \gamma var(\frac{dc}{c})$ consumption CAPM, log too
9. Explain cross sectional and time series variation in average returns, e.g. Fama French Facts. Expected return vs. beta (graph) point of model; variance doesn't matter only covariance or systematic risk;
$$R_t^{ei} = \beta_{i,m}m_t + \varepsilon_t^i$$
$$var(R_t^{ei}) = \beta_{i,m}^2\sigma_m^2 + \sigma_{\varepsilon^i}^2$$
"factor model"
10. MVF; HJ
$$\begin{aligned}\frac{E(R^e)}{\sigma(R^e)} &= -\frac{\sigma(m)\rho(m,R^e)}{E(m)}\\ \|\rho\| &\le 1 \to \frac{\|E(R^e)\|}{\sigma(R^e)} < \frac{\sigma(m)}{E(m)} \approx \gamma\sigma(\Delta c)\end{aligned}$$

(a) $\exists MVF$, HJ bound, frontier reutrns correlated, two fund theorem, Roll theorem. NOT you want to hold mvf

(b) Equity premium risk/free rate puzzles.

$$\begin{aligned}
\frac{\|E(R^e)\|}{\sigma(R^e)} &< \gamma\sigma(\Delta c) \\
\frac{0.08}{0.16} &< \gamma \times 0.02 \rightarrow \gamma > 25 \\
r_t^f &= \delta + \gamma E_t\left(d\log c_t\right) - \frac{1}{2}\gamma^2\sigma_t^2\left(d\log c_t\right)
\end{aligned}$$

$$\begin{aligned}
\frac{E(R^e)}{\sigma(R^e)} &\approx \gamma\sigma(\Delta c)\rho \\
\frac{0.08}{0.16} &\approx \gamma \times 0.02 \times (\rho < 0.5) \rightarrow \gamma > 50
\end{aligned}$$

$$\begin{aligned}
r_t^f &= \delta + \gamma E_t\left(d\log c_t\right) - \frac{1}{2}\gamma^2\sigma_t^2\left(d\log c_t\right) \\
0.01 &= \delta + 25 \times 0.01? \rightarrow \delta = -0.24? \\
0.01 &= 0.01 + \gamma \times 0.01 - \frac{1}{2}\gamma^2 0.02^2 \rightarrow \gamma = 50
\end{aligned}$$

$$\frac{\partial r_t^f}{\partial E_t \Delta c_{t+1}} = 50?$$

History. β, γ to match

$$\begin{aligned}
0 &= E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t+1}^e\right] \\
1 &= E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_t^f\right]
\end{aligned}$$

(c) Continuous time expression

$$\begin{aligned}
dR_t &= \left(r_t^f + \mu_R\right)dt + \sigma_R dz_t^R \\
\frac{d\Lambda_t}{\Lambda_t} &= -r_t^f dt + \sigma_\Lambda dz_t^\Lambda
\end{aligned}$$

$$\begin{aligned}
\mu_R &= \sigma_R\sigma_\Lambda\rho(dz^R dz^\Lambda) \\
\frac{\|\mu_R\|}{\sigma_R} &\leq \sigma_\Lambda
\end{aligned}$$

Note both sides are $\sqrt{dt}$

11. Radom walks and martingales.

$$0 = E_t \left[d \left(\Lambda_t V_t \right) \right] \ \Lambda_{t+1} V_{t+1} = \Lambda_t V_t + \varepsilon_{t+1}$$

12. "Efficiency": price incorporates information. "Joint hypothesis theorem." Implications: information sets $0 = E \left[d \left(\Lambda_t V_t \right) | \text{prices} \right];\ \ 0 = E \left[d \left(\Lambda_t V_t \right) | \text{public} \right]\ 0 = E \left[d \left(\Lambda_t V_t \right) | \text{inside} \right]$. Why was it important? *an organizing framework for empirical work.* Example: trading rules. Fund managers. Announcement effects.

13. General equlibrium:

    (a) Endowment economy

    (b) Full linear economy. Classic finance

    $$dR_t = \mu dt + \sigma dz_t$$

    →portfolio theory→aggregation→CAPM. Qantities are endogenous, we derive the composition of the market portfolio

    (c) Halfway 1: PIH model. R^f is nailed down and the allocation of consumption *over time*, but not across states

    (d) Halfway 2: Q theory, adjustment costs to risky investment projects

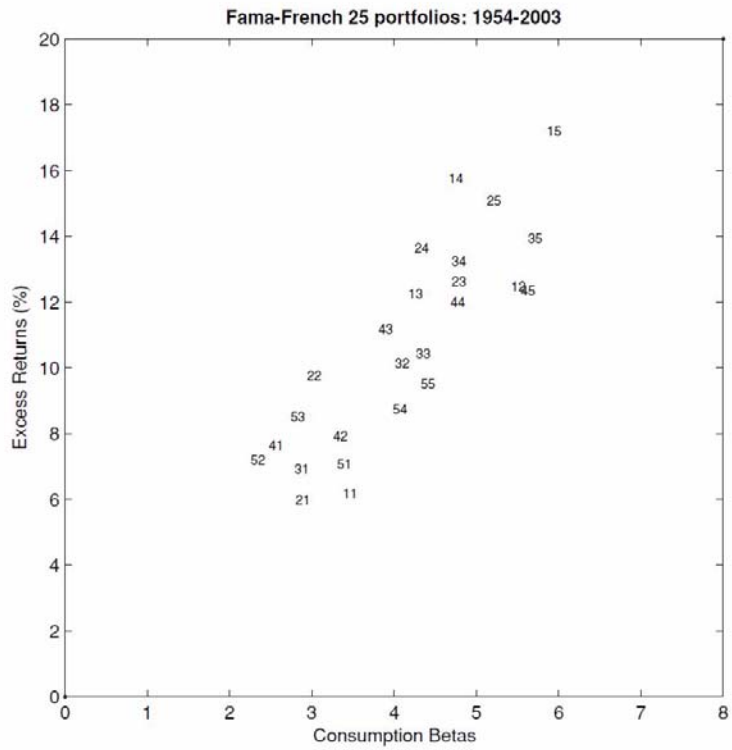

Figure 1. Annual excess returns and consumption betas. This figure plots the average annual excess returns on the 25 Fama–French portfolios and their consumption betas. Each two-digit number represents one portfolio. The first digit refers to the size quintile (1 smallest, 5 largest), and the second digit refers to the book-to-market quintile (1 lowest, 5 highest).

Table II

Annual Excess Returns and Consumption Betas

Panel A reports average annual excess returns on the 25 Fama–French portfolios from 1954 to 2003. Annual excess return is calculated from January to December in real terms. All returns are annual percentages. Panel B reports these portfolios' consumption betas estimated by the time-series regression:

$$R_{i,t} = \alpha_i + \beta_{i,c}\Delta c_t + \varepsilon_{i,t},$$

where $R_{i,t}$ is the excess return over the risk-free rate, and Δc_t is Q4-Q4 consumption growth calculated using fourth quarter consumption data. Panel C reports t-values associated with consumption betas.

	Low		Book-to-market		High
		Panel A: Average Annual Excess Returns (%)			
Small	6.19	12.47	12.24	15.75	17.19
	5.99	9.76	12.62	13.65	15.07
Size	6.93	10.14	10.43	13.23	13.94
	7.65	7.91	11.18	12.00	12.35
Big	7.08	7.19	8.52	8.75	9.50
		Panel B: Consumption Betas			
Small	3.46	5.51	4.26	4.75	5.94
	2.89	3.03	4.79	4.33	5.21
Size	2.88	4.10	4.35	4.79	5.71
	2.57	3.35	3.90	4.77	5.63
Big	3.39	2.34	2.83	4.07	4.41
		Panel C: t-values			
Small	0.93	1.71	1.59	1.83	2.08
	0.98	1.27	2.02	1.83	2.10
Size	1.15	1.93	2.17	2.07	2.39
	1.14	1.75	1.90	2.26	2.39
Big	1.71	1.32	1.67	2.15	2.00

But...

1. t stats on betas are low. This regression *does not* test the model: ER^e. vs. β tests the model. But it means betas are poorly measured, possibly sensitive to a very few data points. (? worth checking in the data!)
2. $E(R^e) = \beta \times (\gamma\sigma(\Delta c))$ very large γ is still implied
3. Does not check riskfree rate implications – the heart of the equity premium puzzle
4. The correct nonlinear version works much worse! It should be $1 = E(\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R)$, but raising things to the 50 power makes it work much worse