10 Week 3 Asset Pricing Theory Extras

1. From $p = E(mx)$ to all of asset pricing. Everything we do is just special cases, that are useful in various circumstances.

   (a) In most of finance we do not use consumption data. We instead use other tricks to come up with an m that works better in practical applications.

   (b) **Theorem:** *If there are no arbitrage opportunities, then we can find an m with which we can represent prices and payoffs by* $p = E(mx)$

   (c) Thus, the m structure allows us to do "no arbitrage" asset pricing.

   (d) From $E(R^i) = R^f + \beta_{i,\Delta c}\lambda_{\Delta c}$ to CAPM, multifactor models, APT, etc.

      i. Basic idea: We can't see Δc. So, we proxy $\Delta c = a + bR^m$, consumption goes down when the market goes down. $\rightarrow CAPM$.

      ii. Is that it? Do *other* things drive changes in consumption? (The CAPM isn't just the statement that consumption goes down when the market goes down; it's the statement that consumption *only* goes down when the market goes down.) $\Delta c = a + b_1 R^m + b_2 X \rightarrow E(R)$ depends on $cov(R, R^m)$ and $cov(R, X)$ This leads to multifactor models

   (e) Bond prices

$$\begin{aligned} P_t^{(1)} &= E_t(m_{t+1} \times 1) \\ P_t^{(2)} &= E_t(m_{t+1} m_{t+2} \times 1). \end{aligned}$$

   Term structure models (Cox Ingersoll, Ross, etc.): model m_{t+1}, (model Δc_{t+1}),

      i. For example

$$m_{t+1} = \phi m_t + \varepsilon_{t+1}$$

      Then

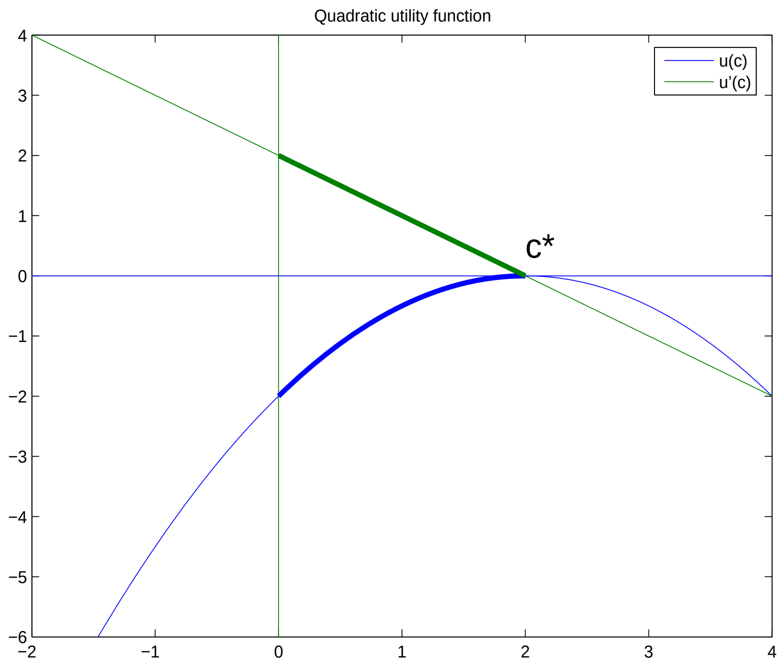
$$\begin{aligned} P_t^{(1)} &= \phi m_t \\ P_t^{(2)} &= \phi^2 m_t = \phi P_t^{(1)} \\ P_t^{(3)} &= \phi^3 m_t = \phi^2 P_t^{(1)} \\ P_t^{(N)} &= \phi^N m_t = \phi^{N-1} P_t^{(1)} \end{aligned}$$

      Look! We have a "one-factor arbitrage-free" model of the term structure. We can draw a smooth curve through bond prices (and then yields) in a way that we know does not allow arbitrage. (Week 8)

   (f) Option pricing (Black-Scholes). Rather than price options from consumption, find m that prices stock and bond, then use that m to price option. (*Asset Pricing* derivation of Black-Scholes)

2. Quadratic utility is very popular (it lies behind mean-variance frontiers). It's only an approximation, though easy to work with.

$$u(c) = -\frac{1}{2}(c^* - c)^2 \rightarrow u'(c) = (c^* - c) \quad (c < c^*)$$



Quadratic utility makes deriving the CAPM easy, and mean-variance portfolio theory.

3. An example of why *co*variance is important. Suppose there are two states u, d tomorrow with probability 1/2 (As in binomial option pricing.)

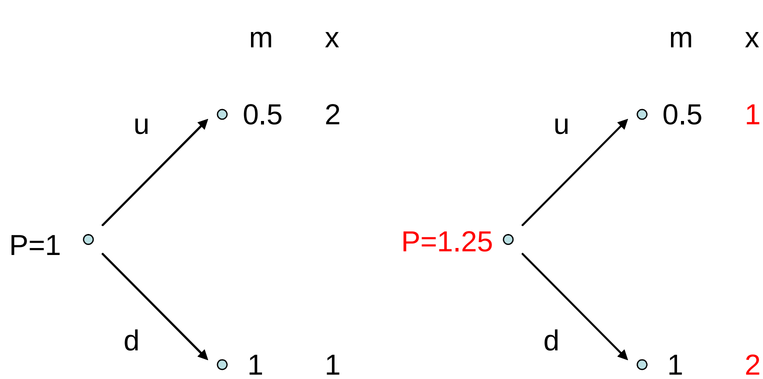
$$p_t = E(mx) = \frac{1}{2}m_u x_u + \frac{1}{2}m_d x_d.$$

u is "good times" with high c, low m. Thus, suppose $m_u = 0.5$, $m_d = 1$. Now, suppose x pays off well in "good times", If $x_u = 2$, $x_d = 1$.

$$p_t = E(mx) = \frac{1}{2} \times 0.5 \times 2 + \frac{1}{2} \times 1 \times 1 = 1.$$

But suppose we switch – same volatility but x pays off well in bad times and badly in good times. $x_u = 1, x_d = 2$.

$$p_t = E(mx) = \frac{1}{2} \times 0.5 \times 1 + \frac{1}{2} \times 1 \times 2 = 1.25.$$



Note $E(x)$, $\sigma(x)$ is the same. The payoff is worth more if the good outcome happens when m is high (hungry) rather than when m is low (full). The *same* m and the *same* x deliver *different* risk adjustments depending on $cov(m, x)$.

4. "Risk-neutral pricing". How $p = E(mx)$ is the same as what you learned in options/fixed income classes.

   (a) Our formula

$$p = E(mx) = \sum_{s=1}^{S} \pi_s m_s x_s$$

   (b) "Risk-neutral probabilities" (Veronesi, options pricing) Define

$$\begin{aligned} p &= \sum_s \pi_s m_s x_s \\ &= \left(\sum_s \pi_s m_s\right) \sum_s \frac{\pi_s m_s}{\left(\sum_s \pi_s m_s\right)} x_s \\ &= \frac{1}{R^f} \sum_{s=1}^{S} \pi_s^* x_s \\ p &= \frac{1}{R^f} E^*(x) \end{aligned}$$

   if we define

$$R^f \equiv \frac{1}{E(m)} = \frac{1}{\sum_s \pi_s m_s}.$$

   and

$$\pi_s^* = \frac{\pi_s m_s}{\sum_s \pi_s m_s} = R^f \pi_s m_s$$

   (we'll see $R^f = 1/E(m)$ below; for now just use it as a definition)

   i. Note

$$\sum_s \pi_s^* = 1$$

   so they could be probabilities[10].

   ii. Interpretation of $p = \frac{1}{R^f} E^*(x)$: price equals risk-neutral expected value using special "risk-neutral probabilities" π^*

   (c) *A discount factor m is the same thing as a set of "risk neutral probabilities"*

   i. Option 1: find "probabilities" π^* that price stock and bond using $p = \frac{1}{R^f} E^*(x)$. Use those probabilities to price option using the same formula

   ii. Option 2: find m that prices stock and bond using $p = E(mx)$. Use that m to price option using $p = E(mx)$. (Using true probabilities)

   iii. These are exactly the same thing!

5. Beta model (CAPM) reminders:

   (a) The steps of running a beta model:

[10] Also since m comes from $u'(c)$ and $u'(c) > 0$, $\pi_s^* > 0$ which probabilities have to obey

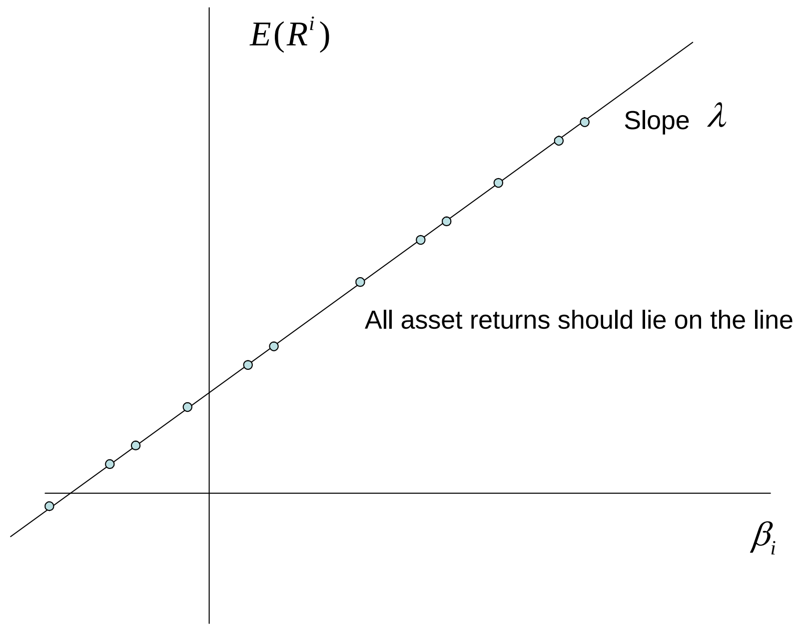
i. Run *time series regression* to find betas

$$R^i_{t+1} = a_i + \beta_{i,\Delta c}\Delta c_{t+1} + \varepsilon^i_{t+1} \;\; t = -1, 2, ...T \text{ for each } i$$

ii. Average returns should be linearly related to betas,

$$E(R^i) = R^f + \beta_{i,\Delta c}\lambda_{\Delta c}$$

β is the *right hand variable* (x), λ is the *slope coefficient* (β)



(b) i in R^i to emphasize

i. The answer to FF question: *This is about why average returns of one asset are higher than of another (cross section). NOT about fluctuation in ex-post return (why did the market go up yesterday?) or predicting returns (will the market go up tomorrow?)*

ii. ER^i, β^i, vary across assets; "quantity of risk". λ is common to all. "price of risk."

(c) Is high $E(R^e)$ good or bad?

i. Neither. An asset must offer high $E(R^e)$ (good) to compensate investors for high risk (bad).

ii. This is about *equilibrium*, *after* the market has settled down, *after* everyone has made all their trades. It's about $E(R)$ that *will last*, not disappear as soon as investors spot it.

iii. Example: what if we all want to short? The price must fall until we're happy to hold the market portfolio again. How must price and $E(R)$ adjust so that people are happy to hold assets?

6. Long lived securities, the explicit derivation:

$$U = E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

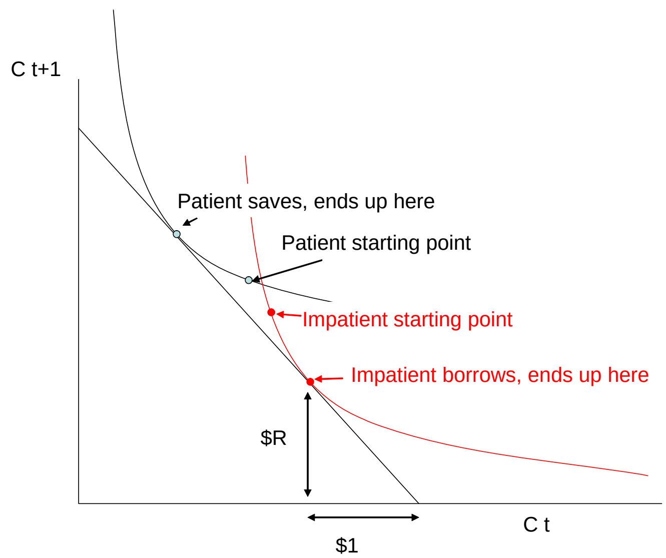
pay $p_t\xi$, get $\xi d_{t+1}, \xi d_{t+2}...$

$$p_t u'(c_t) = E_t \sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) d_{t+j}$$
$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j}$$
$$p_t = E_t \sum_{j=1}^{\infty} m_{t,t+j} d_{t+j} = E_t \sum_{j=1}^{\infty} (m_{t+1} m_{t+2} ... m_{t+j}) d_{t+j}$$

This is the present value formula with *stochastic discount factor.*

7. Question: What if people have different γ, β, or different utilities? Then we get different prices depending on who we ask?

   Answer: Yes if we're asking about genuinely new securities (then sell to the highest value guy). But no if we are talking about market prices. In a market everyone adjusts until they value things at the margin the same way. Example: One is patient, prefers consumption later. One investor is impatient, prefers consumption now. At their starting point, the patient investor implies a lower interest rate, as you would expect. But facing the same market rate, P saves more and I borrows more, until *at the margin* they are willing to substitute over time at the same interest rate, as shown.



8. Question: you slipped in to talking about economy-wide average consumption, not individual consumption. What's up with that?

   Answer: Right. There is a "theory of aggregation" that lets us do this. Here's what needs to be proved: that the average consumption across people responds to market prices just as if there is a single consumer with "average" risk aversion γ and discount rate β doing the choosing. Under some assumptions, it's true. This is natural – in thinking about "high interest rates got people to save more" we don't obviously have to talk about some people being different than others; we can take first cut at the problem by thinking about the behavior of a "representative person."

9. Multifactor models, the explicit vector version

$$E(R^{ei}) = \beta_{i,f^1}\lambda_1 + \beta_{i,f^2}\lambda_2 + ...\text{or } E(R^{ei}) = \beta_i'\lambda$$

where β are (usually) defined from *multiple* regressions,

$$\begin{aligned} R^{ei}_{t+1} &= \alpha + \beta_{i,f^1} f^1_{t+1} + \beta_{i,f^2} f^2_{t+1} + ... + \varepsilon^i_{t+1};\ t = 1,2,...T \\ \text{or } R^{ei}_{t+1} &= \alpha + \beta_i' f_{t+1} + \varepsilon^i_{t+1} \end{aligned}$$

$$\beta = \begin{bmatrix} \beta_{i,f^1} \\ \beta_{i,f^2} \\ \vdots \\ \beta_{i,f^n} \end{bmatrix};\ f_{t+1} \begin{bmatrix} f^1_{t+1} \\ f^2_{t+1} \\ \vdots \\ f^n_{t+1} \end{bmatrix}$$

10. Algebra Fact: multifactor models are equivalent to linear models for m,

$$\begin{aligned} E(R^{ei}) &= \beta_i'\lambda \Leftrightarrow 0 = E_t(m_{t+1}R^{ei}_{t+1}); \\ m_{t+1} &= a - b_1 f^1_{t+1} - b_2 f^2_{t+1} + ... \\ \text{or } m_{t+1} &= a - b' f_{t+1} \end{aligned}$$

(a) The algebra for the equivalence is easy for a single factor

$$\begin{aligned} 0 &= E(mR^{ei}) = E(m)E(R^{ei}) + cov(m, R^{ei}) \\ E(R^{ei}) &= -R^f cov(R^{ei}, m) \\ &= -R^f cov(R^{ei}, a - bf) \\ &= cov(R^{ei}, f)\left(R^f b\right) = \beta_{i,f}\lambda_f \end{aligned}$$

(b) Example 1, last time

$$m_{t+1} = 1 - \delta - \gamma\Delta c_{t+1} \Leftrightarrow E(R^{ei}) = \beta_{i,\Delta c}\lambda_{\Delta c}.$$

(c) Example 2, CAPM

$$m_{t+1} = a - bR^{em}_{t+1} \Leftrightarrow E(R^{ei}_{t+1}) = \beta_{R^{ei}_{t+1}, R^{em}_{t+1}}\lambda$$

(d) *Point: If we can justify $m = a - bf$, we have a multifactor model $E(R^e) = \beta'\lambda$, no need to do the algebra again and again.*

11. The real APT with multiple assets. You can be smarter. How about the Sharpe ratio from clever *portfolios*. For example, if two assets are perfectly negatively correlated, $\alpha^i/\sigma(\varepsilon^i)$ may be small for each one, but if you have 1/2 of each, you have zero risk. The real APT: The maximum Sharpe ratio available in a cleverly chosen *portfolio* of many R^{ei} should be small. This is also a useful formula in general: How do you find optimal portfolios!

(a) Background: A set of returns R^e with covariance matrix Ω. What is the portfolio that gives the best Sharpe ratio? Answer:

$$w = \text{constant} \times \Omega^{-1}E(R^e)$$

We will only get an answer up to scale of course, since $2\times R^{ep}$ has the same SR as R^{ep}. What is the SR of the best portfolio? Answer: $SR_{max} = \sqrt{E(R^e)'\Omega^{-1}E(R^e)}$

(b) Derivation (good practice with matrices!)

$$R^e = \begin{bmatrix} R^{e1}_{r+1} \\ R^{e2}_{t+1} \\ \vdots \\ R^{eN}_{t+1} \end{bmatrix}; \ \Omega = \begin{bmatrix} \sigma^2(R^{e1}) & cov(R^{e1}, R^{e2}) & cov(R^{e1}, R^{e3}) & \cdot \\ & \sigma^2(R^{e2}) & cov(R^{e2} R^{e3}) & \cdot \\ & & \sigma^2(R^{e3}) & \cdot \\ & & & \sigma^2(R^{eN}) \end{bmatrix}; w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

Portfolios

$$R^p = \sum_{i=1}^{N} w_i R^{ei} = w' R^e;$$

Problem:

$$\max_{\{w\}} \frac{E(R^p)}{\sigma(R^p)}$$

This is the same as

$$\min_{\{w\}} \sigma^2(R^p) \text{ given } E(R^p) = \mu,$$

the mean-variance frontier.

$$\begin{aligned} \sigma^2(R^p) &= var(w'R^e) = w'\Omega w \\ E(R^p) &= E(w'R^e) = w'E(R^e) \end{aligned}$$

$$\min_{\{w\}} w'\Omega w - \lambda w' E(R^e)$$

$$\begin{aligned} \Omega w &= \lambda E(R^e) \\ w &= \lambda \Omega^{-1} E(R^e) \end{aligned}$$

Thus we have, *Answer 1: Optimal portfolio.*

(c) We were here to find Sharpe ratios,

$$\frac{E(R^p)}{\sigma(R^p)} = \frac{w'E(R^e)}{\sqrt{w'\Omega w}} = \frac{\lambda E(R^e)'\Omega^{-1}E(R^e)}{\sqrt{\lambda^2 E(R^e)'\Omega^{-1}E(R^e)}} = \sqrt{E(R^e)'\Omega^{-1}E(R^e)}$$

Answer 2: Max SR from these assets is $\sqrt{E(R^e)'\Omega^{-1}E(R^e)}$

(d) Now, what about our multifactor model? Start with a regression

$$R^e_{t+1} = \alpha + \beta f_{t+1} + \varepsilon_{t+1}$$

where

$$R^e_{t+1} = \begin{bmatrix} R^{e1}_{r+1} \\ R^{e2}_{t+1} \\ \vdots \\ R^{eN}_{t+1} \end{bmatrix}; \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}; \varepsilon_{t+1} = \begin{bmatrix} \varepsilon^1_{r+1} \\ \varepsilon^2_{t+1} \\ \vdots \\ \varepsilon^N_{t+1} \end{bmatrix};$$

i.e. FF 25 test assets, FF 3 factors. Form a portfolio of the assets, possibly hedged with factors.

$$R^{ep}_{t+1} = w' R^e_{t+1} - v' f_{t+1}$$

Solve the problem: max Sharpe ratio of such portfolios

$$\max_{\{w,v\}} \frac{E\left(R^{ep}\right)}{\sigma\left(R^{ep}\right)}$$

Answer:

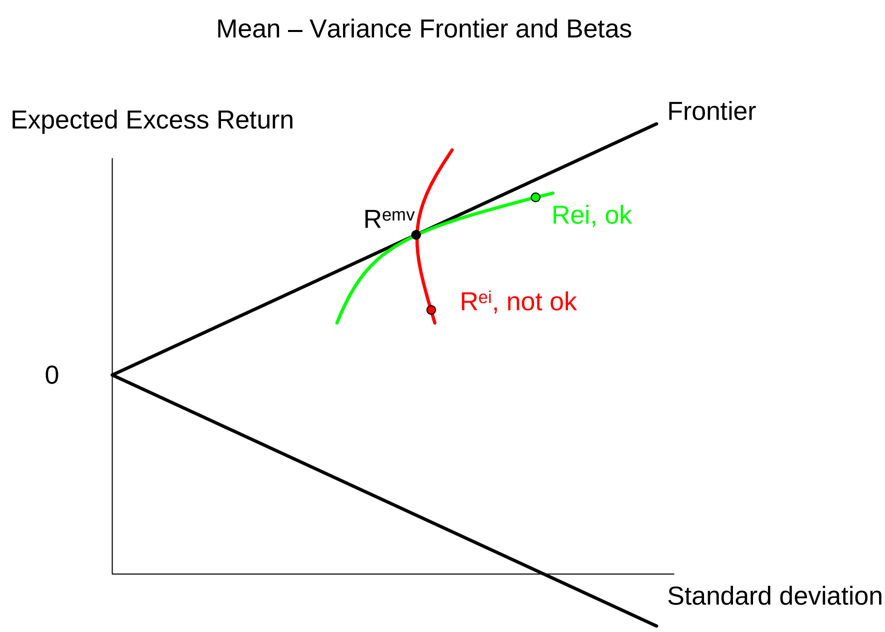
$$SR^2 = \underbrace{E(f)'cov(f,f')^{-1}E(f)}_{\text{max } SR \text{ from factors alone}} + \underbrace{\alpha' cov(\varepsilon,\varepsilon')^{-1}\alpha}_{\text{extra SR from exploiting } \alpha}$$

Interpretation: You can buy any portfolio of f, and you can get any portfolio of $(\alpha + \varepsilon)$ by buying R^e and selling f. Now, f and ε are uncorrelated so the problem separates.

(e) *Conclusion:* For traders: here is the SR you can get from investing. If the boss says "market neutral" then the second term is the max SR you can get from your alpha machine.

(f) *Conclusion:* For economists. When the traders are done, $\alpha' cov(\varepsilon,\varepsilon')^{-1}\alpha$ should be reasonable. If $cov(\varepsilon,\varepsilon')$ is small, so should alpha.

12. A direct proof that mean-variance efficiency implies a single-factor model, 35000 style. reminder. Suppose R^{emv} is on the mean-variance frontier, meaning it has maximum Sharpe ratio. Suppose you form a portfolio that shades a bit in the direction of a particular security[11], i.e. $R^{ep} = R^{emv} + \varepsilon R^{ei}$. If R^{emv} is on the mean-variance frontier, then this move must have the same Sharpe ratio as the R^{emv} portfolio (the green R^{ei}, ok case) If it increased the Sharpe ratio, then, the original portfolio was not on the mvf. If it decreased the Sharpe ratio, then going in the other direction, shorting R^e would increase the Sharpe ratio (the Rei, not ok case). See the drawing.



Let's figure out the change in mean and standard deviation of your portfolio from adding a

[11]Portfolio weights don't have to add to one here, since these are excess returns.

very small ε

$$\begin{aligned} E\left(R^{ep}\right) &= E\left(R^{emv}\right) + \varepsilon E\left(R^{ei}\right) \\ \frac{dE\left(R^{ep}\right)}{d\varepsilon} &= E\left(R^{ei}\right) \end{aligned}$$

$$\begin{aligned} \sigma(R^{ep}) &= \sqrt{\sigma^2\left(R^{emv}\right) + \varepsilon^2\sigma^2\left(R^{ei}\right) + 2\varepsilon cov(R^{emv}, R^{ei})} \\ \frac{d\sigma(R^{ep})}{d\varepsilon} &= \frac{1}{2}\left(\cdot\right)^{-\frac{1}{2}} \times \left[2\varepsilon\sigma^2\left(R^{ei}\right) + 2cov(R^{emv}, R^{ei})\right] \\ \left.\frac{d\sigma(R^{ep})}{d\varepsilon}\right|_{\varepsilon=0} &= \frac{cov(R^{emv}, R^{ei})}{\sigma(R^{emv})} = \beta_{i,R^{emv}}\sigma(R^{emv}) \end{aligned}$$

(Words: If you add a small amount of R^{ei} to the portfolio R^{emv}, the volatility of your *portfolio* goes up by $\beta_{i,R^{emv}}\sigma(R^{emv})$. $\sigma(R^{ei})$ does not matter!)

(a) Now, if we're going to have the "ok" case from the drawing, it must be that

$$\begin{aligned} \frac{dE\left(R^{ep}\right)}{d\varepsilon} &= \frac{E(R^{emv})}{\sigma(R^{emv})}\left.\frac{d\sigma(R^{ep})}{d\varepsilon}\right|_{\varepsilon=0} \\ E\left(R^{ei}\right) &= \frac{E(R^{emv})}{\sigma(R^{emv})}\beta_{i,R^{emv}}\sigma(R^{emv}) \\ E\left(R^{ei}\right) &= \beta_{i,R^{emv}}E(R^{emv}) \end{aligned}$$

11 Equity Premium

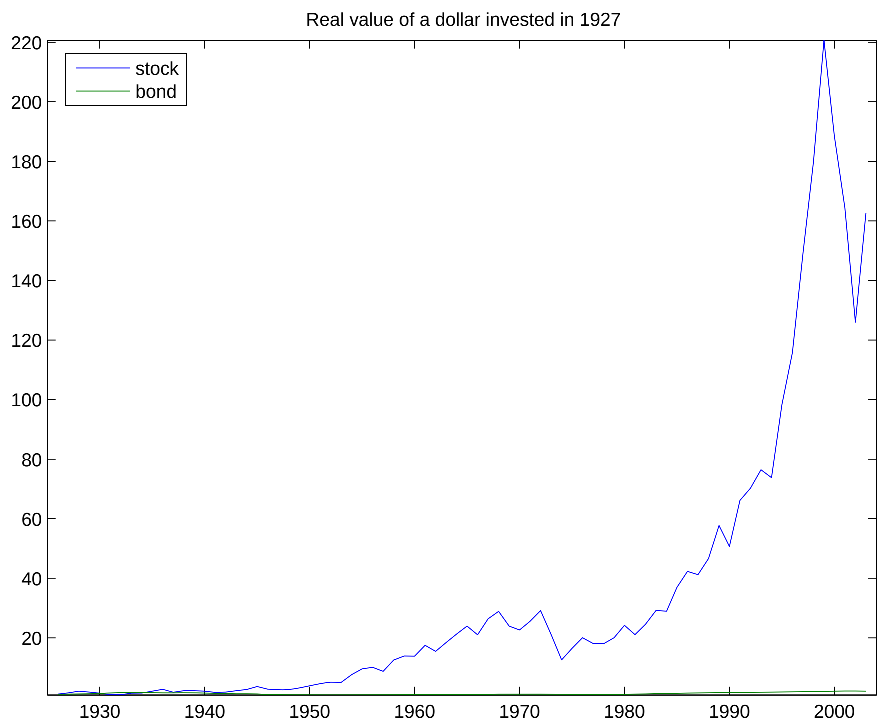
11.1 Equity Premium and Macroeconomic Risk overheads

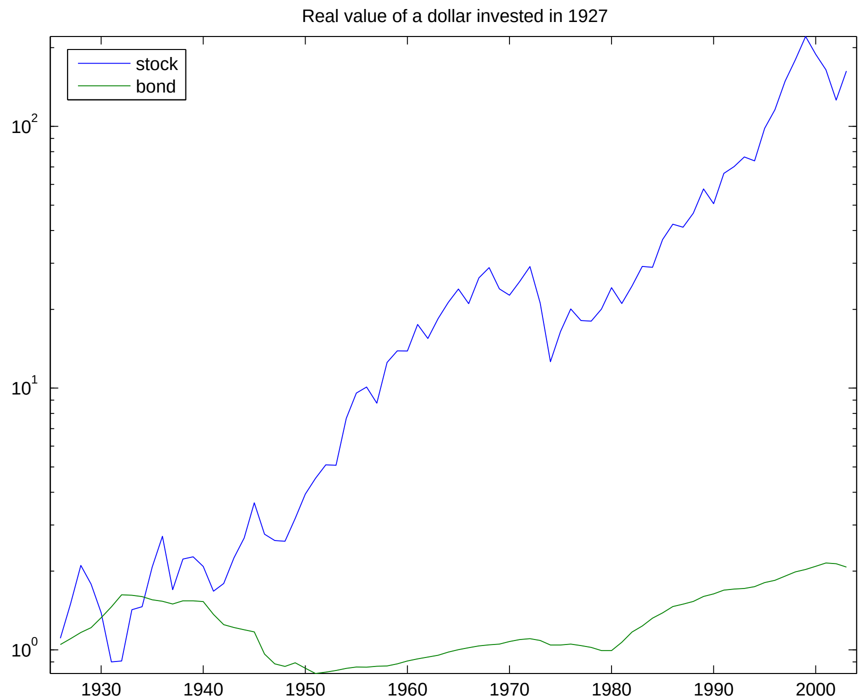
Annual data 1948-2003, percent

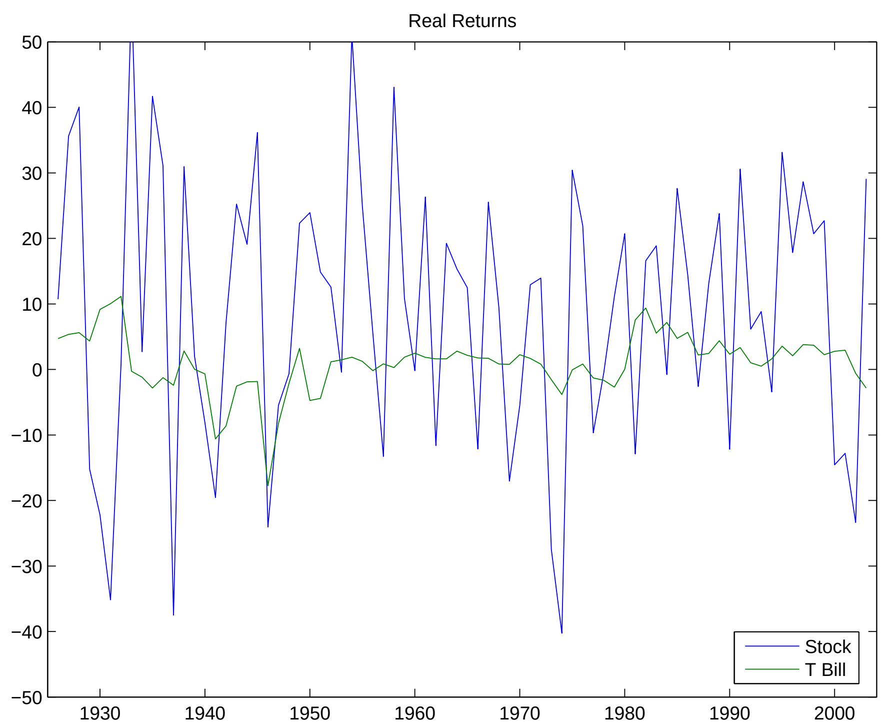
$E(\Delta c)$	$\sigma(\Delta c)$	$E(R^e)$	$\sigma(R^e)$	$E(R^{\text{bond,real}})$	$\text{corr}(\Delta c, R^e)$
1.33	1.92	7.70	18.0	1.6	0.41

Value of $1 invested

Horizon (Years)	Stock	Bond
5	1.51	1.05
10	2.29	1.12
20	5.22	1.25
30	11.94	1.40
50	62.44	1.76

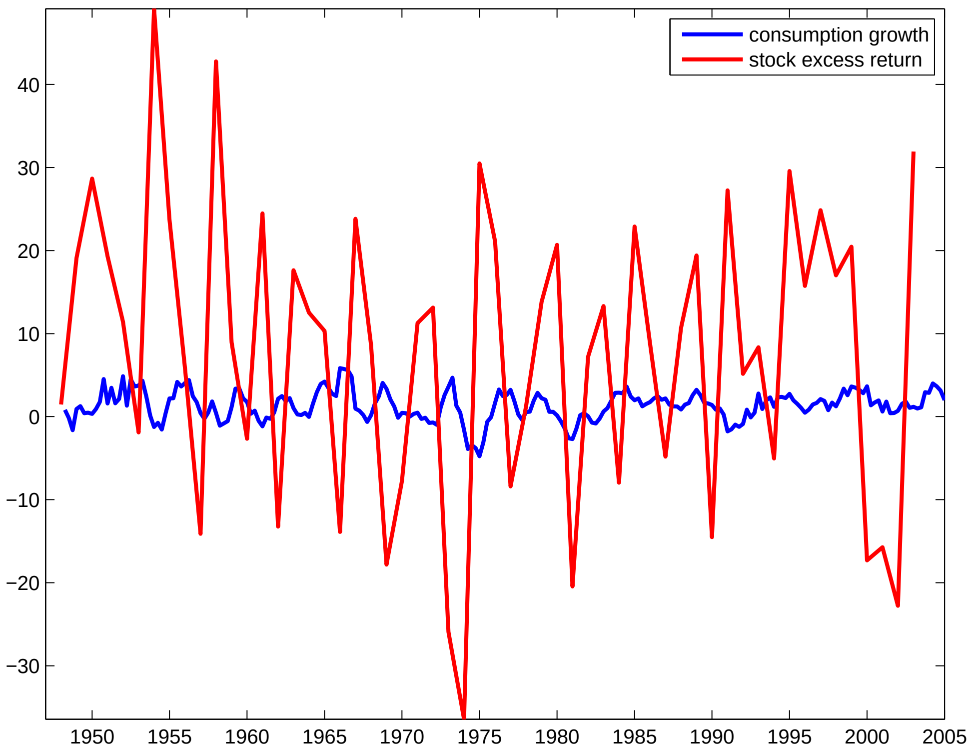






Annual data 1948-2003, percent

$E(\Delta c)$	$\sigma(\Delta c)$	$E(R^e)$	$\sigma(R^e)$	$E(R^{\text{bond,real}})$	$\text{corr}(\Delta c, R^e)$	$\text{cov}(R^e, \Delta c)$
1.33	1.92	7.70	18.0	1.6	0.41	14.09



	1927-2002 Stock-TB	TB
Mean	7.49	1.13
Std dev	20.9	4.40
Std. error $\sigma/\sqrt{T}$	2.38	0.50
Mean +/- 1 σ (66%)	$5.11 - 9.87$	
Mean +/- 2 σ (95%)	2.73− 12.25	

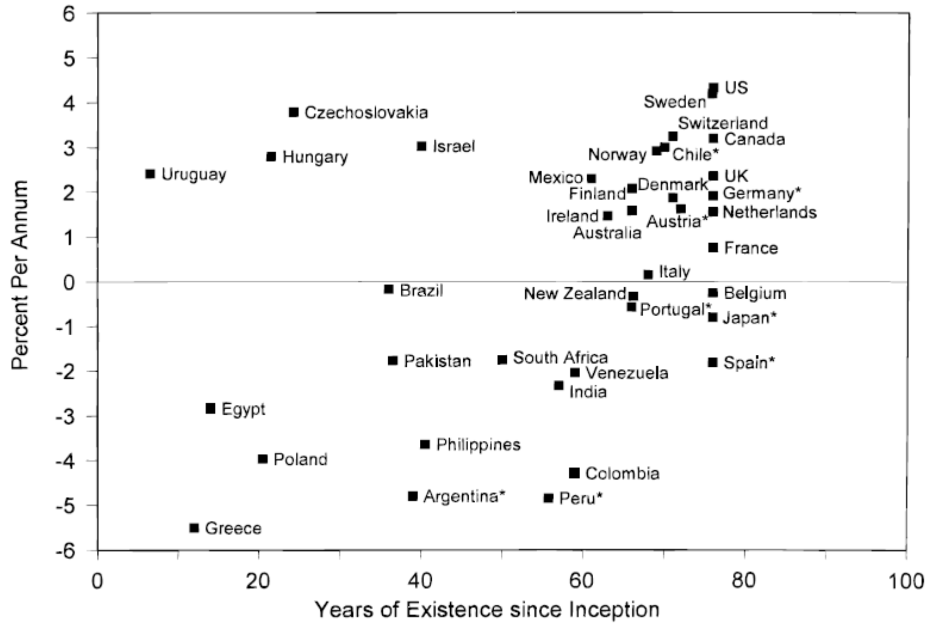
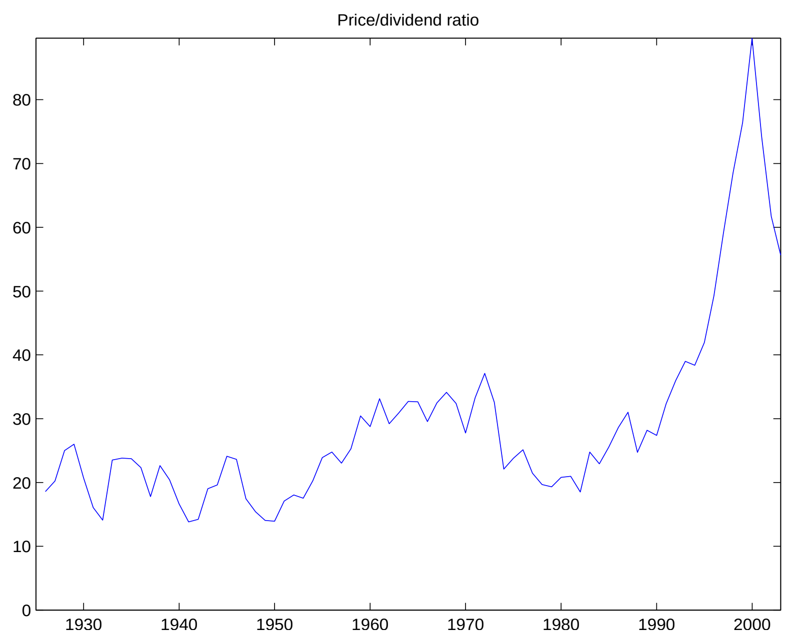


Figure 1. Real returns on global stock markets. The figure displays average real returns for 39 markets over the period 1921 to 1996. Markets are sorted by years of existence. The graph shows that markets with long histories typically have higher returns. An asterisk indicates that the market suffered a long-term break.



	Real Stock	**Premium**	Real Bond
1927-2003	8.87	7.84	1.04
1927-1982	8.36	7.88	0.48
1983-2003	10.35	7.70	2.65

11.2 Equity Premium and Macroeconomic Risk

1. Q: What *is* the expected return on the market portfolio?
2. *Unconditional* (50^o in Chicago) vs, *conditional* mean – given today's high P/E, P/D, etc. (20^o next week given Jan, cold this week)?
3. This is *the* central number.
   (a) CAPM $E(R^{ei}) = \beta_{im} E(R^{em})$. But what's $E(R^{em})$?
   (b) Cost of capital. Do we build a factory?
   (c) Investors, social security. Stock/bond allocation?
4. CAPM, FF3F do not answer this question. $E(R^{em})$ is an *input* to CAPM (then FF3F, i.e. what are $E(hml)$, $E(smb)$?)
5. Equity premium puzzle: we can use our simple models to try to understand the equity premium.
6. Historical averages.

Annual data 1948-2003, percent

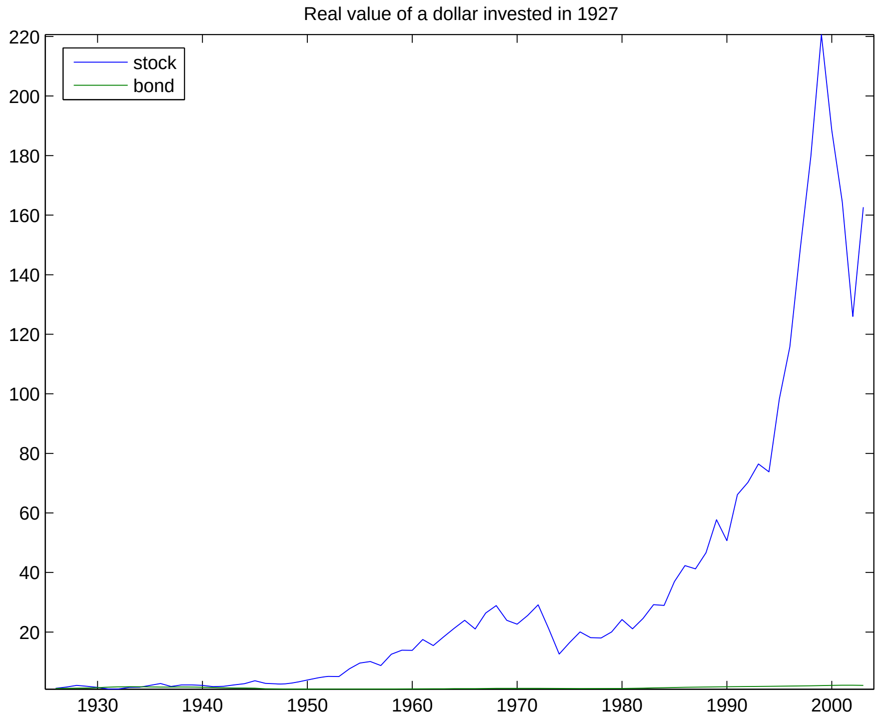
$E(\Delta c)$	$\sigma(\Delta c)$	$E(R^e)$	$\sigma(R^e)$	$E(R^{\text{bond,real}})$	$\text{corr}(\Delta c, R^e)$
1.33	1.92	7.70	18.0	1.6	0.41

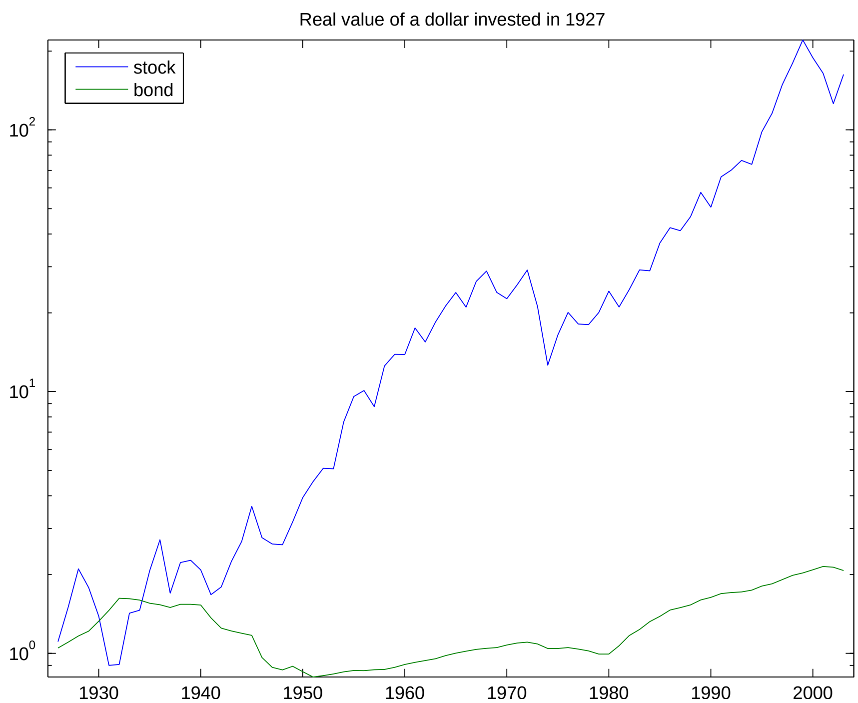
   (a) 6% - 7.5% is the conventional wisdom. Do we believe it? Will it last?
   (b) No 100 year secrets. Hence, *Is 7.5% a believable compensation for risk? Are people happy* **not** *buying more stocks given a 7.5% premium?* If the conclusion is "we all should buy more stocks," the premium will disappear.
7. *7.5% is huge.* It's hard to believe people would not buy more for a 7.5% premium.
   (a)

Value of $1 invested

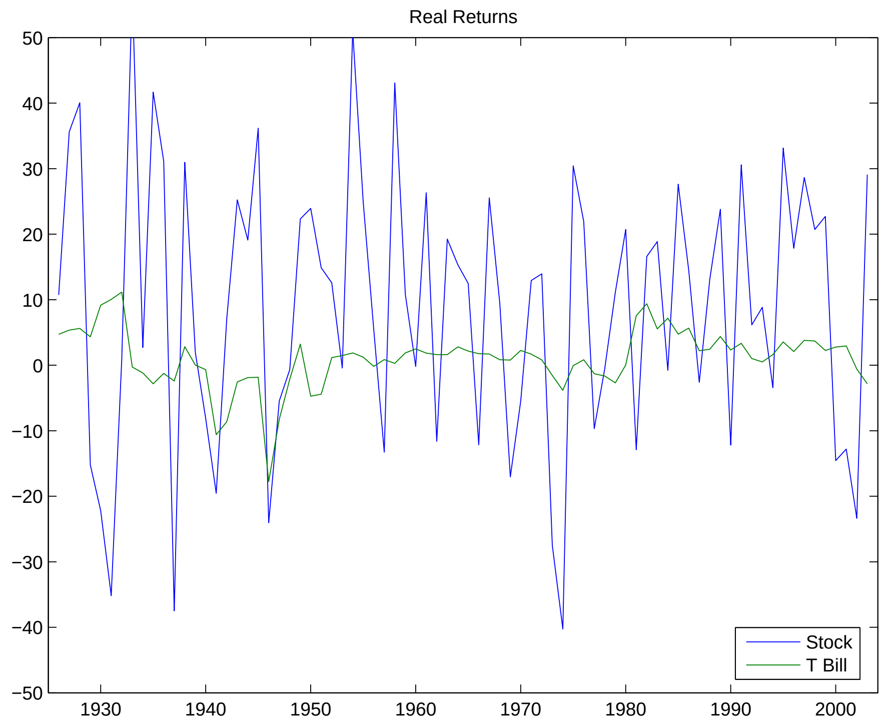
Horizon (Years)	Stock	Bond
5	1.51	1.05
10	2.29	1.12
20	5.22	1.25
30	11.94	1.40
50	62.44	1.76

(b)





8. Risk is real too of course. Look closely at last figure – generation-long losses are possible (63-83) Or,



9. Implications:

   (a) Trader: "ER? Let's buy!"

   (b) Economist (cautious trader) ER? Especially for 100 years? There must be some risk keeping everyone out.

   (c) Economist-trader: If this was "let's buy", will it be there for the future now that everyone knows about it?

10. Is the risk enough to keep people from wanting more? Back to theory.

$$E(R^e_{t+1}) \approx \gamma cov(\Delta c_{t+1}, R^e_{t+1})$$

Reminder: By focusing on the *premium,* "Growth of US economy" "savings" "boomers saving for retirement" etc. are irrelevant to the *difference* between stocks and bonds. It's about *allocation* of savings to stock vs. bonds, not about the *level* of saving. Finally

$$\begin{aligned} E(R^e_{t+1}) &\approx \gamma\sigma(\Delta c_{t+1})\sigma(R^e_{t+1})\rho(R^e, \Delta c) \\ \text{market sharpe ratio } \frac{E(R^e_{t+1})}{\sigma(R^e_{t+1})} &\approx \gamma\sigma(\Delta c_{t+1})\rho(R^e, \Delta c) \end{aligned}$$

11. Facts, again

Annual data 1948-2003, percent

$E(\Delta c)$	$\sigma(\Delta c)$	$E(R^e)$	$\sigma(R^e)$	$E(R^{\text{bond,real}})$	corr$(\Delta c, R^e)$	cov$(R^e, \Delta c)$
1.33	1.92	7.70	18.0	1.6	0.41	14.09

(a) Consumption growth *is* correlated with stock returns – stocks go down in bad times (0.41) and so should offer a return premium. Theory provides a *qualitative* (story-telling) explanation.

(b) But consumption growth is a lot smoother than stock returns, and this causes trouble for a *quantitative* explanation.

$$\begin{aligned}\frac{E(R^e_{t+1})}{\sigma(R^e_{t+1})} &\approx \gamma\sigma(\Delta c_{t+1})\rho(R^e,\Delta c)\\ \frac{7.7}{18.0} &= 0.43 \approx \gamma \times 0.0192 \times 0.41\\ \gamma &\approx \frac{7.7}{18.0}\frac{1}{0.0192 \times 0.41} = 54.3!\end{aligned}$$

(c) Even if $\rho = 1$, our mean-variance frontier formula shows up.

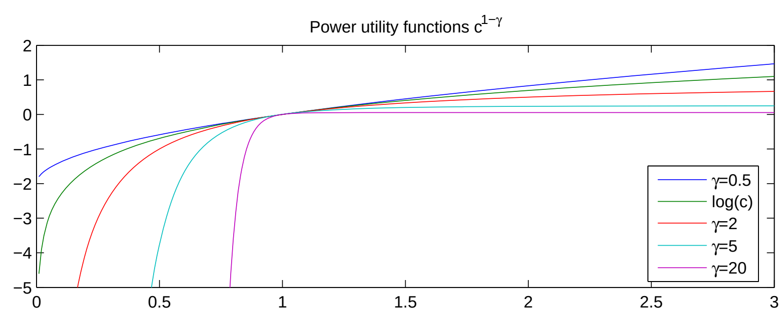
$$\text{market sharpe } \frac{\left\|E(R^e_{t+1})\right\|}{\sigma(R^e_{t+1})} \leq \gamma\sigma(\Delta c_{t+1})$$

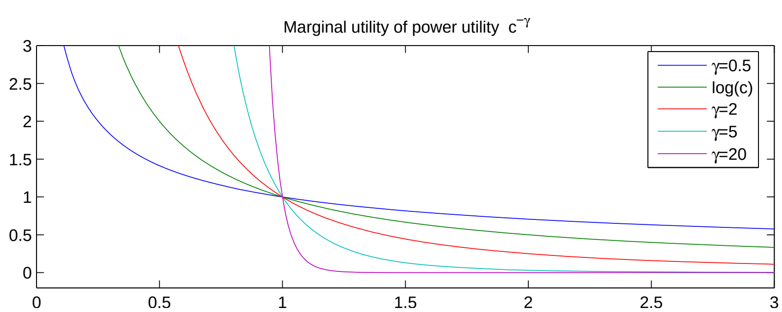
(Stop and look at what we've done – an equation bounding the market sharpe ratio based on the *riskiness* and *risk aversion* of the economy.)

$$\begin{aligned}0.43 &\leq \gamma \times 0.0192\\ \frac{0.43}{0.0192} &= 22.4 \leq \gamma!\end{aligned}$$

12. $\gamma = 22.4$ is *HUGE.* 54 is bigger!

(a)





(b)

$	Willingness to pay to avoid a bet, if consumption is \$50k/year						
				γ			
bet	0	0.5	1	2	5	10	50
5.00	0	0.00	0.00	0.00	0.00	0.00	0.01
50.00	0	0.01	0.02	0.05	0.12	0.25	1.25
500.00	0	1.25	2.50	5.00	12.50	24.97	120.25
5000.00	0	125.31	250.88	500.00	1217.00	2211.92	4358.96

50 doesn't look so bad for a \$5 or even \$50 bet, but totally weird for a \$500 or \$5000 bet. (Warning: distrust survey/experimental evidence! – how the question is asked makes a big difference).

(c) What we really want is detailed evidence on risk aversion from real economic decisions. Alas it's missing. Gambling. Extreme Motocross. Extended warranties on \$40 DVD players.

13. Even if you accept large γ, it causes a *risk-free rate puzzle.* Recall

$$r_t^f \approx \delta + \gamma E_t \left(\Delta c_{t+1}\right)$$

(a) If γ is huge (50) then

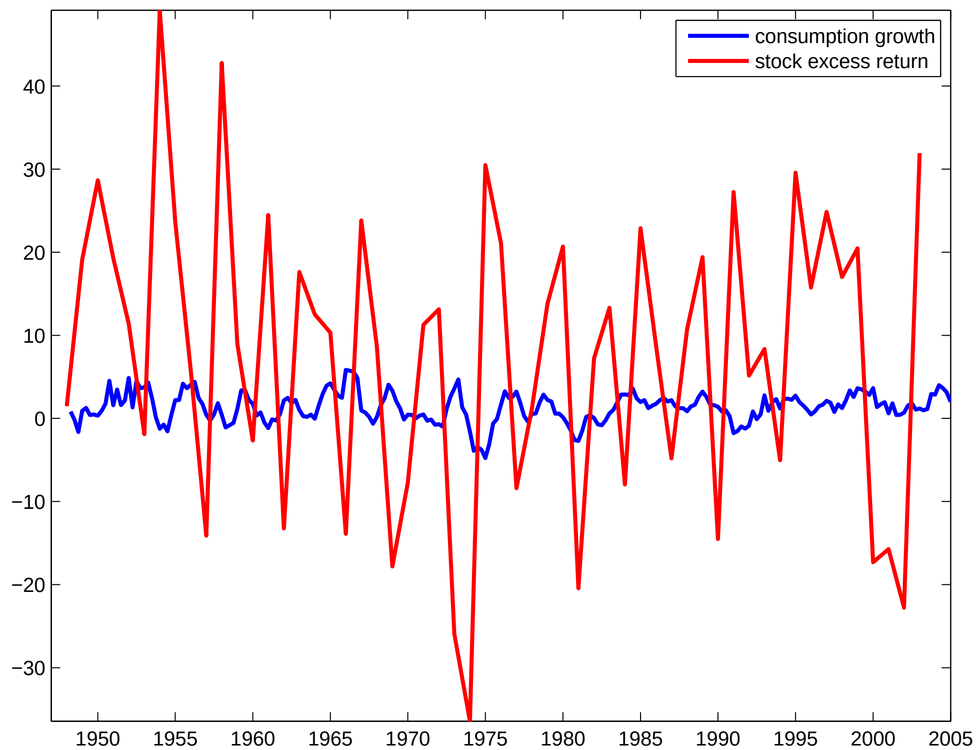
$$\begin{aligned} 1\% &= \delta + \gamma \times (1.33\%) \\ 1\% &= \delta + 50 \times (1.33\%) \\ 1\% &= \delta + 66.5\% \\ \delta &= -65.5\% \end{aligned}$$

People *prefer the future by 66%???* This is nuts.

(b) Worse, $\gamma = 50$ means that a 1% increase in $E(\Delta c)$ (coming out of a recession) implies a 50% (percentage point) rise in interest rates!!! We see nothing like this.

(c) Technical Solution to allow high risk aversion without interest rate problems: new utility functions that distinguishes *intertemporal substitution* from *risk aversion.* Say people are very willing to (say) put off buying a car for a year if they can save at 6% rather than 5% interest rates, but almost completely unwilling to invest in stocks that give 1% worse payout in recession and 1% better in expansion. But do we believe this?

14. The source of the problem: consumption is very smooth.



(a) Consumption and stocks *do* move together in the very long run though. If stocks are down 50% in 2020, so will consumption. "Consumption ignores temporary stock price fluctuation." Does this give us a hope for an answer in long run data? This is a current research topic.

(b) Why did the CAPM not notice this problem? The CAPM has a hidden assumption $C_{t+1}/C_t = R^m_{t+1}$. If consumption were as volatile as market returns, $\sigma(\Delta c) = \sigma(R^m) = 18\%$, and $\rho = 1$, there would be no problem.

$$\begin{aligned} \frac{\|E(R^e_{t+1})\|}{\sigma(R^e_{t+1})} &\leq \gamma\sigma(\Delta c_{t+1}) \\ 0.43 &\leq \gamma \times 0.18 \\ \frac{0.43}{0.18} &= 2.4 \leq \gamma! \end{aligned}$$

Traditional CAPM and portfolio theory with $\gamma = 2 - 5$ work fine, but implicitly assume $\sigma(\Delta c) = 18\%$! ***All portfolio optimizers have this problem!***

(c) Note what counts is nondurables, or the flow of enjoyment from durables, not durables purchases.

(d) *Consumption is smooth; our economy is not that risky. Economic theory does not deliver anything like a 7.5% equity premium unless people are very, very risk averse.*

(e) And if they are so incredibly averse to risk – accepting consumption that is different across *states of nature*, why are they so little averse to shifting consumption *over time* – why do small changes in consumption growth not spark huge changes in interest rates?

15. Responses:

   (a) LOTS (me included; see ch. 20).

      i. Q: Individual $\sigma(\Delta c)$ larger than economy average?
      A1: *Nobody* has $\sigma(\Delta c) = 20\%$
      A2: And individual $\sigma(\Delta c)$ are less correlated with stock returns so this doesn't help

$$\begin{aligned} \frac{E(R^e_{t+1})}{\sigma(R^e_{t+1})} &\approx \gamma\frac{cov(\Delta c, R^e)}{\sigma(R^e)} = \gamma\frac{cov(\Delta c, R^e)}{\sigma^2(R^e)}\sigma(R^e) = \gamma\sigma\left(\beta_{\Delta c, R^e}R^e\right) \\ \Delta c_{t+1} &= \beta_{\Delta c, R^e}R^e_{t+1} + v_{t+1} \end{aligned}$$

      Only the $\beta_{\Delta c, R^e}R^e_{t+1}$ component matters – and that's not uncorrelated across people

      ii. Q: Not everyone holds stocks?
      A: Still a puzzle for those who do. Does *stockholder's* consumption vary enough? And rich people do most consumption too!

      iii. Q: Different utility functions?
      A: It's not really the *shape* that matters, as we're really talking about the second derivative of the utility function. To change things, you need *other arguments*, for example that yesterday's consumption, or today's labor changes the marginal utility of consumption, $\partial u(c, x)/\partial c$. So far, these do not avoid high risk aversion (see long surveys, e.g. by me.)

   (b) Result: Extremely high risk aversion has not yet been avoided if you want to produce 7.5% mean return / 0.5 Sharpe ratio

   (c) In a nutshell: why do people fear stock market risk so much, and gambling so little? A hint of the answer: stock market losses come *in unpleasant states of the world.* But what are those?

16. If it makes no economic sense, is it really there?

   (a) Lots of statistical uncertainty. Despite a 70 year average, but the *volatility* of stocks means we don't know much about the *mean.* It's hard to measure something that's jumping up and down.

	1927-2002	
	Stock-TB	TB
Mean	7.49	1.13
Std dev	20.9	4.40
Std. error $\sigma/\sqrt{T}$	2.38	0.50
Mean +/- 1 σ (66%)	5.11 – 9.87	
Mean +/- 2 σ (95%)	2.73– 12.25	

   (b) $\sigma/\sqrt{T}$ and stocks more generally.

      i. Using 1 year: $\sigma/\sqrt{T} = 18\%$. Can't measure 8%!

      ii. Using 5 years: $\sigma/\sqrt{T} = 18/\sqrt{5} = 8.05\%$. 8% is only 1σ. 5 years of data are useless for mean returns of something like a stock.

iii. Using 20 years $\sigma/\sqrt{T} = 18/\sqrt{20} = 4.02\%$ The bare minimum for measuring 8% returns

Stock returns are so volatile that measuring mean returns is very hard, even with a century of data.

(c) (But.. if tracking error is small, you can often measure performance *relative to an index*, even if you don't know whether the *index itself* does well. Hence, α can often be well measured, when R^2 is high. This happened in the FF3F regressions. This is one reason funds are held to tracking error constraints.

$$\begin{aligned} R_t^{ei} &= \alpha_i + \beta_i f_t + \varepsilon_t^i \\ \sigma(\hat{\alpha}) &= \sigma(\varepsilon)/\sqrt{T} \end{aligned}$$

If the R^2 is high, $\sigma(\varepsilon) << \sigma(R^e)$. You can accurately measure the *difference* between two highly correlated variables.)

(d) Beyond standard errors: *Selection bias and Rare Events.* There is no Russian, Ugandan equity premium 1900-2000. Like hedge funds, maybe stocks are all about once per century crashes which will bring the "true mean" back down to something reasonable.

From Jorion and Goetzmann, 1999, "Global Stock Markets in the 20th Century" *Journal of Finance* 54

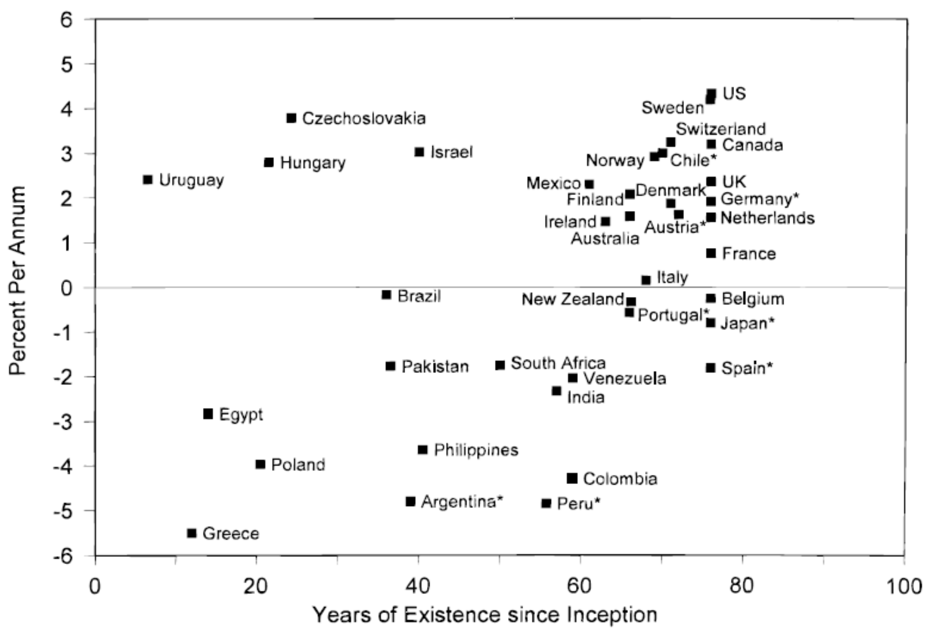
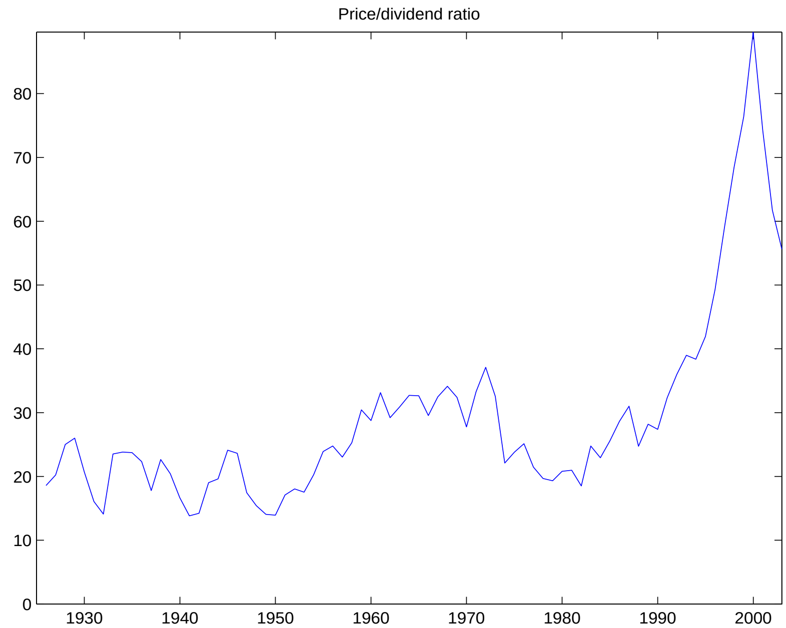


Figure 1. Real returns on global stock markets. The figure displays average real returns for 39 markets over the period 1921 to 1996. Markets are sorted by years of existence. The graph shows that markets with long histories typically have higher returns. An asterisk indicates that the market suffered a long-term break.

17. Is the huge increase in price/x ratio the "good luck" accounting for a spurious equity premium? (Fama and French)



(a) One way to answer this: how were returns to 1982?

	Real Stock	**Premium**	Real Bond
1927-2003	8.87	7.84	1.04
1927-1982	8.36	7.88	0.48
1983-2003	10.35	7.70	2.65

A: this seems not to be the answer. (Note here I'm disagreeing with Fama and French 2000 and the bottom of p. 461 which swallowed their argument a bit too uncritically.)

(b) How much of large return comes from the rise in P/D? Returns come from

i. Dividend yield – I pay \$1, I get 4¢ dividend, that's 4% return.

ii. Dividend growth at current price/dividend ratio – If dividends rise from 4¢ to 5¢ and P/D doesn't change, that means prices go up by 25% too, giving me a 25% return.

iii. Changes in the price/dividend ratio. If P/D rises from 20 to 21, that's a 1/20 = 4% return.

iv. In equations,[12]

$$R_{t+1} \approx 1 + \Delta \left(P/D\right)_{t+1} + \frac{D_t}{P_t} + \Delta D_{t+1}$$

[12]Derivation:

$$\begin{aligned} R_{t+1} &= \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{\left(\frac{P_{t+1}}{D_{t+1}} + 1\right)\frac{D_{t+1}}{D_t}}{\frac{P_t}{D_t}} \\ &= \left(\frac{P_{t+1}/D_{t+1}}{P_t/D_t} + \frac{D_t}{P_t}\right)\frac{D_{t+1}}{D_t} \\ R_{t+1} &\approx 1 + \Delta \left(P/D\right)_{t+1} + \frac{D_t}{P_t} + \Delta D_{t+1} \end{aligned}$$

where

$$\Delta D_{t+1} \equiv \frac{D_{t+1}}{D_t} - 1$$
$$\Delta(P/D)_{t+1} \equiv \frac{P_{t+1}/D_{t+1}}{P_t/D_t} - 1$$

v. Up to 82 – it's not PD so *if it's a surprise, it's a surprise in* ΔD. *The surprise was that* **economic growth** *would be so high.*

vi. Long run stock returns are driven by long run ΔD, once P/D reverts.

18. Bottom line: Did your grandparents really look at the world in 1948, say "Stocks will outperform bonds by 7.5% per year for the rest of the century. But I don't want any more because I am afraid of the risks."? If not, *a good part of the sample 7.5% was luck.* The form of that luck was *higher than expected economic growth.*

    (a) If it was luck, the true, ex-ante premium is lower than the 7.5% we usually use.

    (b) ....And the *conditional* premium (given still high P/D) is even worse!!

19. But... We seem to see large apparent premia in lots of other ways. Value/Growth, Small/Big, Corporate/treasury spreads etc.

$$\frac{\|E(R^e_{t+1})\|}{\sigma(R^e_{t+1})} < \gamma\sigma(\Delta c)$$

applies to *any* Sharpe ratio and *high Sharpe ratios are pervasive in finance.* So maybe it is real.

20. Bigger points:

    (a) A little simple Chicago economics lets you organize your thoughts on *the most important issue* in asset pricing – where will stocks/bonds go in the next 50 years?

    (b) We know a lot less about this number than you thought! Well, Quantifying your ignorance (and everyone else's) is true wisdom.

    (c) In the end, we must tie risk premia to *real*, *macroeconomic* events. If not, they really are just "buy opportunities."

    (d) What is the equity premium? I wish I knew! (My guess 2-3% but we won't know for a long time.)

11.3 Equity Premium Summary/Review

1. Q: What *is* the expected return on the market portfolio?

2. Historical averages. $E(R^e) \approx 8\%$, $\sigma(R^e) \approx 16\%$, Sharpe ≈ 0.5. $E(\Delta c) \approx \sigma(\Delta c) \approx 1 - 2\%$

3. An 8% return premium is HUGE. Risk justifying this reward?

4. From $p = E(mx)$,

$$E(R^e_{t+1}) \approx \gamma cov(\Delta c_{t+1}, R^e_{t+1})$$
$$\text{Sharpe } \frac{E(R^e_{t+1})}{\sigma(R^e_{t+1})} \approx \gamma\sigma(\Delta c_{t+1})\rho(R^e, \Delta c)$$

5. From numbers we need *huge* γ to fit this.
6. Even if you accept huge γ *risk-free rate puzzle.* Huge γ predicts high r^f, or negative δ, and r^f very sensitive to consumption growth

$$r_t^f \approx \delta + \gamma E_t \left(\Delta c_{t+1}\right)$$

7. Source of the problem: consumption risk $\sigma(\Delta c)$ is much less than stock risk $\sigma(R)$. The CAPM didn't notice because it assumes they are the same and never looks at consumption.
8. Responses have not avoided high γ
9. $\sigma/\sqrt{T}$ is a big problem; we don't know much about mean returns.
10. Is 8% good luck – just recent rise in P/X?

    (a) No, high returns before 1980 too.

    (b)

$$R_{t+1} \approx 1 + \Delta\left(P/D\right)_{t+1} + \frac{D_t}{P_t} + \Delta D_{t+1}$$

    If there was a surprise in the equity premium, it was that ΔD, **economic growth** *would be so high*

    (c) JC view: A lot of US economy success *was* good luck, equity premium is less than 8%