13 Week 5 Empirical methods overheads

13.1 Motivation and overview

1. Expected return, beta model, as in CAPM, Fama-French APT.

$$1.\text{TS regression, define } \beta: \ R_t^{ei} = \alpha_i + \beta_i' f_t + \varepsilon_t^i \quad t = 1, 2...T.$$

$$2.\text{Model: } E(R^{ei}) = \beta_i'\lambda \ \ (+\alpha_i)$$

(α_i+) because the point of the model is that α should $= 0$.

2. How do we take this to the data? Objectives:

   (a) *Estimate* parameters. $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$.

   (b) *Standard errors* of parameter estimates. How do $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ vary if you draw new data and try again?

   (c) *Test* the model. Does $E(R^{ei}) = \beta_i'\lambda$? Are the true $\alpha = 0$? Are the $\hat{\alpha}$ that we see due to bad luck rather than real failure of the model?

   (d) *Test* one model *vs. another*. Can we drop a factor, e.g. size?

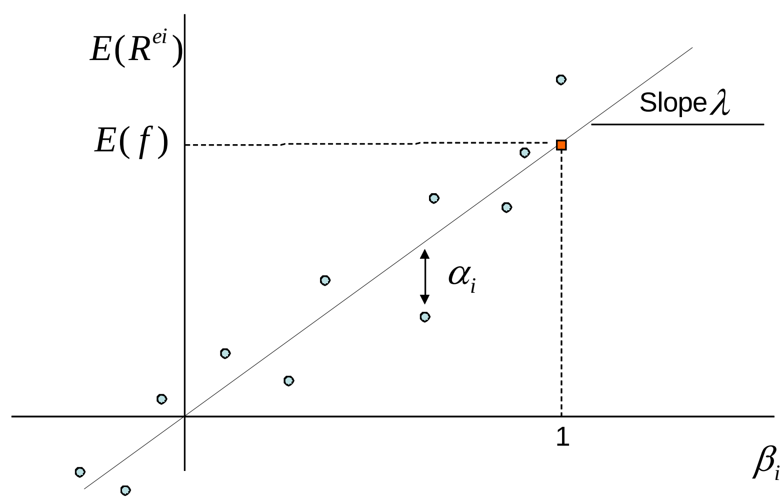
13.2 Time-series regression

1. Time Series Regression (Fama-French)

   (a) *Method:*

      i. Run
      $$R_t^{ei} = \alpha_i + \beta_i' f_t + \varepsilon_t^i \quad t = 1, 2...T. \text{ for each } i$$

      ii. *Interpret* the regression as a description of the cross section, without running more regressions
      $$E\left(R_t^{ei}\right) = \beta_i' E\left(f_t\right) + \alpha_i \ \ i = 1, 2, ...N$$



(b) *Estimates*:

i. $\hat{\alpha}, \hat{\beta}$: OLS time-series regression.

$$R_t^{ei} = \alpha_i + \beta_i' f_t + \varepsilon_t^i \quad t = 1, 2...T \text{ for each } i.$$

ii. $\hat{\lambda}$: Mean of the factor,

$$\hat{\lambda} = \frac{1}{T} \sum_{t=1}^{T} f_t = \bar{f}.$$

(c) *Standard errors:* Assume ε_t^i independent *over time* (but not *across portfolios).*

i. OLS standard errors $\hat{\alpha}_i, \hat{\beta}_i$.

ii. $\hat{\lambda}$:

$$\sigma(\hat{\lambda}) = \frac{\sigma(f_t)}{\sqrt{T}}$$

(d) *Test* α are *jointly* zero?

i. Answer: look at

$$\hat{\alpha}' cov(\hat{\alpha}, \hat{\alpha}')^{-1} \hat{\alpha}.$$

Precise forms,

$$\hat{\alpha}' cov(\hat{\alpha})^{-1} \hat{\alpha} = T \left[1 + \bar{f}' \Sigma_f^{-1} \bar{f}\right]^{-1} \hat{\alpha}' \Sigma^{-1} \hat{\alpha} \,\tilde{}\, \chi_N^2$$

$$\frac{T-N-K}{N} \left[1 + \bar{f}' \hat{\Sigma}_f^{-1} \bar{f}\right]^{-1} \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} \,\tilde{}\, F_{N,T-N-K}$$

where

$$\begin{aligned}
\hat{\alpha} &= \text{ estimated alphas} \\
\bar{f} &= E(f) \\
\Sigma_f &= cov(f, f') \\
\Sigma &= cov(\varepsilon, \varepsilon') \\
T &= \text{sample length} \\
K &= \text{number of factors} \\
N &= \text{number of assets} \\
\chi^2 &= \text{chi-squared distribution} \\
F &= \text{F distribution}
\end{aligned}$$

Do not memorize! Understand, look up.

ii. Basic idea: like the square of a t test

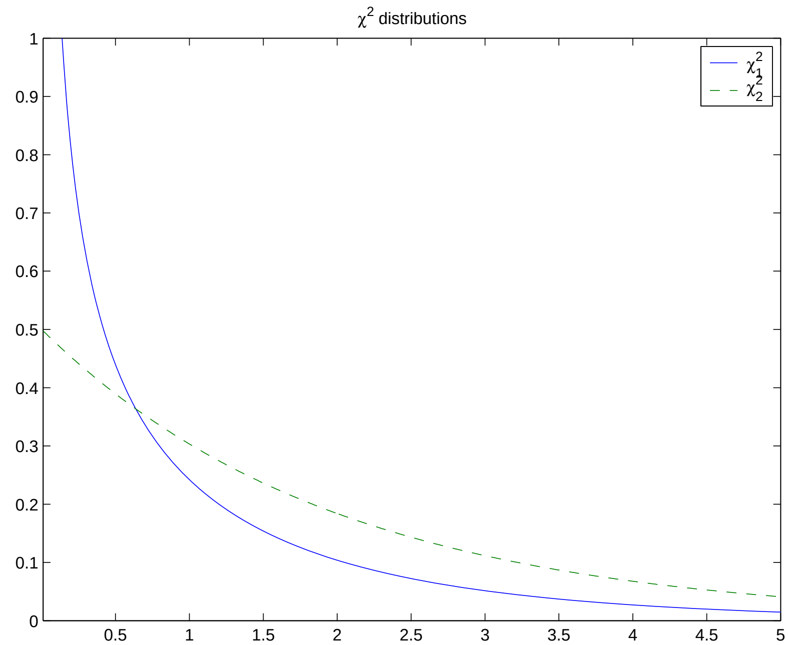
$$\frac{\hat{\alpha}}{\sigma(\hat{\alpha})} \,\tilde{}\, t$$

iii. If ε are also normal then a refinement is valid in small samples.

$$\frac{T-N-K}{N} \left[1 + \bar{f}' \hat{\Sigma}_f^{-1} \bar{f}\right]^{-1} \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha} \,\tilde{}\, F_{N,T-N-K}$$

"GRS Test."

iv. Distribution of $\hat{\alpha}' cov(\hat{\alpha}, \hat{\alpha}')^{-1}\hat{\alpha}$ – if the true alphas are zero, how often should we see this number be one, two, etc.?

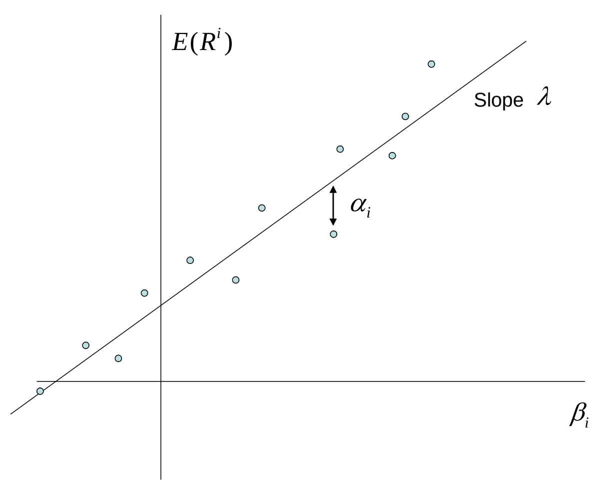


v. Fama and French "GRS" quote p. 57. The huge rejection because Σ is so small.

$$\frac{T-N-K}{N}\left[1+\bar{f}'\hat{\Sigma}_f^{-1}\bar{f}\right]^{-1}\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}\tilde{\ }F_{N,T-N-K}$$

13.3 Cross-sectional regression

1. Another idea. The main point of the model is *across assets,*



$$E(R^{ei}) = \beta_i'\lambda \quad (+\alpha_i) \quad i = 1, 2, ...N$$

Why not fit this as a *cross sectional regression*?

2. Two step procedure

   (a) TS (over time for each asset) to get β_i,

   $$R_t^{ei} = a_i + \beta_i f_t + \varepsilon_t^i \quad t = 1, 2...T \text{ for each } i.$$

   (b) Run CS (across assets) to get λ.

   $$E(R^{ei}) = (\gamma) + \beta_i \lambda + \alpha_i \quad i = 1, 2, ...N$$

3. *Estimates*:

   (a) $\hat{\beta}$ from TS.

   (b) $\hat{\lambda}$ slope coefficient in CS.

   (c) $\hat{\alpha}$ from error in CS: $\hat{\alpha} = \frac{1}{T}\left(\sum_{t=1}^{T} R_t^e\right) - \hat{\lambda}\hat{\beta}$. $\hat{\alpha} \neq a$ *is not the intercept from the time series regression any more.*

4. *Standard errors.*

   (a) $\sigma(\hat{\beta})$ from TS, OLS formulas.

   (b) $\sigma(\hat{\lambda})$. You can't use OLS formulas.

   (c) Answer: With no intercept in CS,

   $$\sigma^2(\hat{\lambda}) = \frac{1}{T}\left[(\beta'\beta)^{-1}\beta'\Sigma\beta(\beta'\beta)^{-1}\left(1 + \lambda'\Sigma_f^{-1}\lambda\right) + \Sigma_f\right]$$

   (d) $cov(\hat{\alpha})$

   $$cov(\hat{\alpha}) = \frac{1}{T}\left(I - \beta(\beta'\beta)^{-1}\beta'\right)\Sigma\left(I - \beta(\beta'\beta)^{-1}\beta'\right)\left(1 + \lambda'\Sigma_f^{-1}\lambda\right)$$

   (e) See notes for the formula with an intercept in CS.

5. *Test*

   $$\hat{\alpha}' cov(\hat{\alpha}, \hat{\alpha}')^{-1}\hat{\alpha}\,\tilde{}\,\chi^2_{N-K-1}.$$

6. General tool: how to correct OLS standard errors for correlated errors

   $$\begin{aligned} Y &= Xb + u;\ E(u\ u') = \Omega \\ \hat{b} &= (X'X)^{-1}X'Y \\ \hat{b} &= (X'X)^{-1}X'(Xb + u) \\ E(\hat{b}) &= b!\text{ (unbiased)} \\ cov(\hat{b}) &= E\left[(X'X)^{-1}X'u\ u'X(X'X)^{-1}\right] \\ &= (X'X)^{-1}X'\Omega X(X'X)^{-1} \end{aligned}$$

   If $\Omega = \sigma_\varepsilon^2$ then we're back to our friend

   $$cov(\hat{b}) = \sigma^2 (X'X)^{-1}$$

7. Point: understand the question these formulas are answering, a bit of why they look so bad (and why OLS formulas are wrong/inadequate.) Know where to go look them up.

13.4 Fama - MacBeth procedure.

1. Run TS to get betas.

$$R_t^{ei} = a_i + \beta_i' f_t + \varepsilon_t^i \quad t = 1, 2...T \text{ for each } i.$$

2. Run a cross sectional regression *at each time period*,

$$R_t^{ei} = (\gamma_t) + \beta_i' \lambda_t + \alpha_{it} \;\; i = 1, 2, ...N \;\; \text{for each } t.$$

3. *Estimates* of λ, α are the *averages across time*

$$\hat{\lambda} = \frac{1}{T} \sum_{t=1}^{T} \hat{\lambda}_t; \;\; \hat{\alpha}_i = \frac{1}{T} \sum_{t=1}^{T} \hat{\alpha}_{it}$$

4. *Standard errors* use our friend $\sigma^2(\bar{x}) = \sigma^2(x)/T$ *but applied to the monthly cs regression coefficients!*

$$\begin{aligned} \sigma^2(\hat{\lambda}) &= \frac{1}{T} var(\hat{\lambda}_t) = \frac{1}{T^2} \sum_{t=1}^{T} \left(\hat{\lambda}_t - \hat{\lambda} \right)^2 \\ cov(\hat{\alpha}) &= \frac{1}{T} cov(\hat{\alpha}_t) = \frac{1}{T^2} \sum_{t=1}^{T} (\hat{\alpha}_{it} - \hat{\alpha}_i)(\hat{\alpha}_{jt} - \hat{\alpha}_j) \end{aligned}$$

   This one main point. These standard errors are easy to calculate.

5. *Test*

$$\hat{\alpha}' cov(\hat{\alpha}, \hat{\alpha}')^{-1} \hat{\alpha} \tilde{\ } \chi^2_{N-1}.$$

6. Fact: if the β are constant over time, the estimates are *identical* to those of cross-sectional regressions! Standard errors are close.

7. Other applications of Fama-MacBeth: any time you have a big cross-section, which may be correlated with each other.

   (a)

$$R_{t+1}^{ei} = a + b \ln size_{it} + c \ln beme_{it} + \varepsilon_{t+1}^i; \quad cov(\varepsilon^i, \varepsilon^j) \neq 0$$

   This is an obvious candidate, since the errors are correlated across stocks but not across time. FF used FMB regressions in the second paper we read.

   (b)

$$\text{investment}_{it} = a + b \times \text{Book/Market}_{it} + c \times \text{profits}_{it} + \varepsilon_{it}$$

   (c) This is a big deal! Either FMB or "Cluster."

13.5 Testing one model vs. another

1. Example. FF3F.
$$E(R^{ei}) = \alpha_i + b_i\lambda_{rmrf} + h_i\lambda_{hml} + s_i\lambda_{smb}$$
Drop size?
$$E(R^{ei}) = \alpha_i + b_i\lambda_{rmrf} + h_i\lambda_{hml}$$

2. See if $\lambda_{smb} = 0$? Common mistake: No, because the b and h change without s.

3. Example 2: CAPM works well for size portfolios, but shows up in FF model
$$E(R^{ei}) = 0 + \beta_i\lambda_{rmrf} \leftarrow \text{ works, } \beta_i \text{ higher with higher } E(R^{ei})$$
$$E(R^{ei}) = 0 + b_i\lambda_{rmrf} + s_i\lambda_{sml} \leftarrow \text{ works too, all } b_i = 1 \text{ and } \lambda_{sml} > 0$$

4. Solution:

   (a) You can drop smb *if the other factors price smb.*
   $$smb_t = \alpha_{smb} + b_s rmrf_t + h_s hml_t + \varepsilon_t$$
   We can drop smb from the three factor model if and only α_{smb} is zero. .

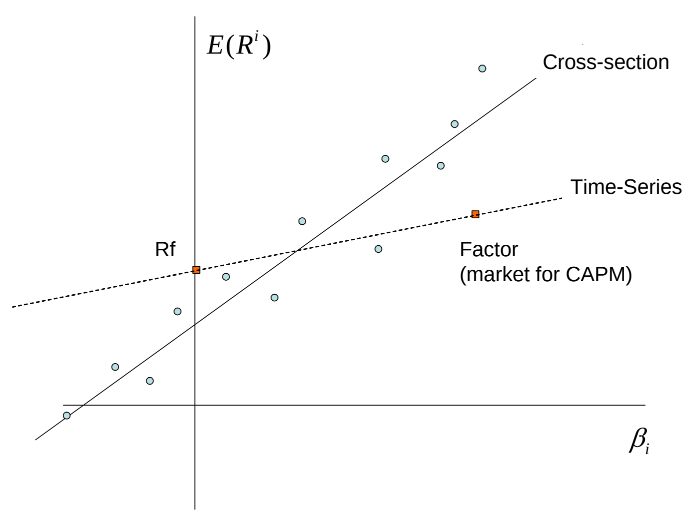
   (b) "Drop smb" means the 25 portfolio alphas are the same with or without smb.

   (c) Equivalently, we are forming an "orthogonalized factor"
   $$smb_t^* = \alpha_{smb} + \varepsilon_t = smb_t - b_s rmrf_t - h_s hml_t$$
   and drop if $E(smb^*) = 0$.

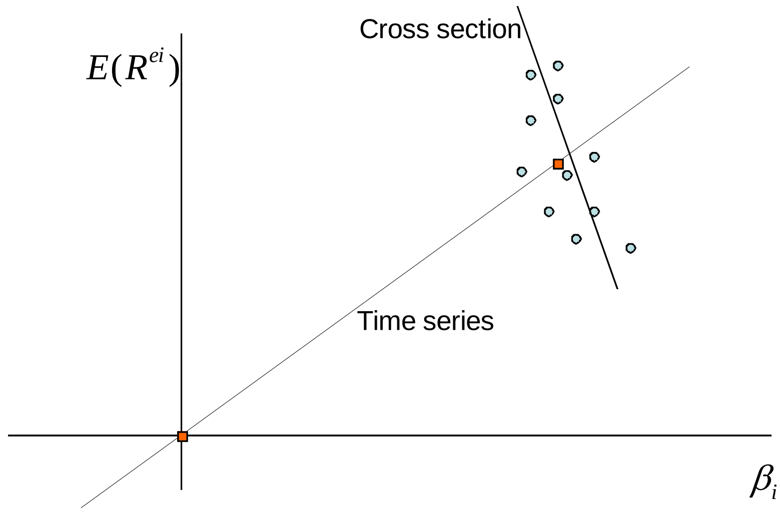
5. Why do FF keep size? It helps explain return *variance*, improves standard errors. Example: 100% R^2 with smb?

13.6 Time series vs. cross section

1. TS vs. CS.



2. Warning:



3. You can't do time series when the factor is not a return → cross section is often used.

$$\text{TS regression:} \qquad R_t^{ei} = \alpha_i + \beta_i' f_t + \varepsilon_t^i$$
$$\text{E}(\,\cdot\,): \qquad E\left(R_t^{ei}\right) = \alpha_i + \beta_i' E\left(f_t\right)$$

   (a) *If the factor is also an excess return* (rmrf, hml, smb), then the model should apply to the factor as well,

$$E\left(R_t^{ei}\right) = \beta_i'\lambda \rightarrow E\left(f\right) = 1 \times \lambda \rightarrow E\left(R_t^{ei}\right) = \beta_i' E(f)$$

   In this case, the α is the mean of a portfolio

$$R^{ei} - \beta_i f = \alpha_i + \varepsilon_i$$

   and in this case *the time series intercept is the same as as the cross-sectional error.*

   (b) *If the factor is not an excess return* you can't do this

$$\text{TS regression:} \qquad R_t^{ei} = a_i + \beta_i' \Delta c_t + \varepsilon_t^i$$
$$\text{E}(\,\cdot\,): \qquad E\left(R_t^{ei}\right) = a_i + \beta_i' E\left(\Delta c_t\right)$$

$$\text{NOT} : E\left(\Delta c_t\right) = 1 \times \lambda$$

   in this case the *time series intercepts a are not the same as the cross sectional errors* α, and *the time series intercept is not a portfolio return,*

$$R_t^{ei} - \beta_i \Delta c_t$$

13.7 Summary of empirical procedures

1. The model is

$$E(R^{ei}) = \beta_i'\lambda + \alpha_i$$

(the α_i should be zero) where the β are defined from time series regressions

$$R_t^{ei} = a_i + \beta_i' f_t + \varepsilon_t^i \quad t = 1, 2, ...T \text{ for each } i.$$

2. *Estimate* parameters, $\hat{\beta}, \hat{\lambda}, \hat{\alpha}$; *standard errors* $\sigma(\hat{\beta}), \sigma(\hat{\lambda}), \sigma(\hat{\alpha})$, an overall *test* $\hat{\alpha}' cov(\hat{\alpha})^{-1}\hat{\alpha}$.

3. Time series regression:

   (a) $\hat{\alpha}, \hat{\beta}, \sigma(\hat{\alpha}), \sigma(\hat{\beta})$ from OLS time series regressions for each asset.

   (b) $\hat{\lambda} = \bar{f}$. $\sigma(\hat{\lambda}) = \sigma(f)/\sqrt{T}$.

   (c) χ^2,F GRS tests for $\hat{\alpha}'\Sigma^{-1}\hat{\alpha}$ to test all α together.

   (d) Note: f must be a return or excess return for this to work. (True for CAPM, FF3F.)

4. Cross sectional regression:

   (a) $\hat{\beta}, \sigma(\hat{\beta})$ from OLS time series regressions for each asset

   (b) $\hat{\gamma}, \hat{\lambda}$ from cross sectional regression

   $$E(R^{ei}) = \gamma + \beta_i'\lambda + \alpha_i. \; i = 1, 2, ..N$$

   Ugly formulas for standard errors $\sigma(\hat{\lambda})$ and $\sigma(\hat{\alpha})$, $cov(\hat{\alpha})$

   (c) $\hat{\alpha}' cov(\hat{\alpha})^{-1}\hat{\alpha}$ to test all α together.

   (d) Can use if f is not a return (Δc for example)

5. Fama -MacBeth

   (a) $\hat{\beta}, \sigma(\hat{\beta})$ from OLS time series regressions for each asset.

   (b) Cross sectional regressions at each time period

   $$R_t^{ei} = \gamma_t + \beta_{ti}'\lambda_t + \alpha_{it} i = 1, 2, ...N \text{ for each } t$$

   Then $\hat{\lambda}$, $\hat{\gamma}$, $\hat{\alpha}$ from averages of $\hat{\lambda}_t$, $\hat{\gamma}_t$, $\hat{\alpha}_t$. Standard errors from our old friend, $\sigma(\hat{\lambda}) = \frac{1}{\sqrt{T}}\sigma(\hat{\lambda}_t)$ similarly for $\hat{\alpha}$

   (c) $\hat{\alpha}' cov(\hat{\alpha})^{-1}\hat{\alpha}$ to test all α together.

13.8 Comments

1. Summary of regressions.

   (a) Forecasting regressions (return on D/P)

   $$R_{t+1} = a + bx_t + \varepsilon_{t+1} \; t = 1, 2, ...T$$

(b) Time series regressions (CAPM)

$$R^i_{t+1} = a_i + \beta_i f_{t+1} + \varepsilon^i_{t+1} \; t = 1, 2, ..T$$

(c) Cross-sectional regressions

$$E\left(R^i_{t+1}\right) = \gamma + \beta_i \lambda + \alpha_i \; i = 1, 2...N$$

(d) (Fama-MacBeth cross-sectional

$$R^i_t = \gamma_t + \beta_i \lambda_t + \alpha_{it} \quad i = 1, 2...N \;)$$

(e) These are *totally different regressions. Don't confuse them. In particular,*

i. TS is *not about "forecasting returns".*

ii. R^2 in TS is not really relevant to the CAPM.

iii. Cross sectional regressions aren't about forecasting returns either.

iv. Only the average value of FMB cross section is really interesting.

2. When is a factor "important"?

$$R^{ei}_t = \alpha_i + b_i rmrf_t + h_i hml_t + \varepsilon_{it}$$

t on h_i and R^2 tells you if hml is important for explaining *variance* of rmrf.

$$hml_t = \alpha_{hml} + \beta rmrf_t + \varepsilon_t$$

$\alpha_{hml} > 0$ tell you if adding hml helps to explain the *mean* $E(R^e)$ of all the other returns

3. When is the TS intercept alpha? *only if the right hand variable is a return*

$$\begin{aligned}
R^{ei}_t &= \alpha_i + b_i rmrf_t + \varepsilon_{it} \;\; YES \\
R^{ei}_t &= \alpha_i + b_i \Delta GDP_t + \varepsilon_{it} \;\; NO \\
R^{ei}_t &= \alpha_i + b_i(rmrf_t > 0) + c_i(rmrf_t < 0) + \varepsilon_{it} \;\; NO \\
R^{ei}_t &= \alpha_i + b_i R^{option}_t + \varepsilon_{it} \;\; YES
\end{aligned}$$