Online Appendix to “The new-Keynesian Liquidity Trap”

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This Appendix collects derivations and formulas for “The new-Keynesian Liquidity Trap.” Computer programs are also available here (the JME website). These materials, and any updates and corrections are also available on my personal website, `http://faculty.chicagobooth.edu/john.cochrane/`

7.1. General solution

Here I derive the general solution (6), (7), (8) . To recap, the model (1), (2) is

$$\frac{dx_t}{dt} = \sigma\left(i_t - r_t - \pi_t\right) \tag{22}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - \kappa(x_t + g_t). \tag{23}$$

I proceed by analogy to discrete-time lag operator methods.

Differentiate (23), and substitute from (22) for dx_t/dt to obtain

$$\frac{d^2\pi_t}{dt^2} - \rho\frac{d\pi_t}{dt} - \kappa\sigma\pi_t = -z_t \equiv -\kappa\sigma(i_t - r_t) - \kappa\frac{dg_t}{dt}. \tag{24}$$

Write this differential equation in the operator form

$$\left(\frac{d}{dt} - \lambda^f\right)\left(\frac{d}{dt} + \lambda^b\right)\pi_t = -z_t. \tag{25}$$

To invert the differential operator (25), note that

$$\left(\frac{d}{dt} - \lambda^f\right)\pi_t = y_t \tag{26}$$

has solution

$$\pi_t = Ce^{\lambda^f t} - \int_{s=t}^{\infty} e^{-\lambda^f(s-t)} y_s ds, \tag{27}$$

while

$$\left(\frac{d}{dt} + \lambda^b\right)\pi_t = y_t \tag{28}$$

has solution

$$\pi_t = Ce^{-\lambda^b t} + \int_{s=-\infty}^{t} e^{-\lambda^b (t-s)} y_s ds. \tag{29}$$

Therefore, write (25) as

$$\pi_t = Ce^{-\lambda^b t} + C_f e^{\lambda^f t} + \frac{1}{\left(\frac{d}{dt} - \lambda^f\right)\left(\frac{d}{dt} + \lambda^b\right)} z_t \tag{30}$$

$$= Ce^{-\lambda^b t} + C_f e^{\lambda^f t} + \frac{1}{\lambda^f + \lambda^b}\left[\frac{1}{\frac{d}{dt} + \lambda^b} - \frac{1}{\frac{d}{dt} - \lambda^f}\right] z_t. \tag{31}$$

Set to zero the forward-explosive solutions $C_f e^{\lambda^f t}$, and we immediately have the solution (6),

$$\pi_t = Ce^{-\lambda^b t} + \frac{1}{\lambda^f + \lambda^b}\left[\int_{s=-\infty}^{t} e^{-\lambda^b (t-s)} z_s ds + \int_{s=t}^{\infty} e^{-\lambda^f (s-t)} z_s ds\right]. \tag{32}$$

We can find the solutions for x_t similarly, or more easily by solving (23) for x_t and differentiating (32). The result is (8), i.e.

$$\kappa x_t = -\kappa g_t + \lambda^f Ce^{-\lambda^b t} + \frac{1}{\lambda^f + \lambda^b}\left[\lambda^f \int_{s=-\infty}^{t} e^{-\lambda^b (t-s)} z_s ds - \lambda^b \int_{s=t}^{\infty} e^{-\lambda^f (s-t)} z_s ds\right]. \tag{33}$$

7.2. Formulas for step function impulses

For $r_t = -r$, $g_t = g$, $i_t = 0$, $T_l < t < T_h$ and $r_t = r$, $g_t = 0$, $i_t = r$ otherwise, evaluating the integrals in (6) and (8), repeated above as (32) and (33), yields

$t \leq T_l$:

$$\pi_t = Ce^{-\lambda^b t} + \frac{\kappa}{\lambda^f + \lambda^b}\left[e^{-\lambda^f (T_l - t)} - e^{-\lambda^f (T_h - t)}\right]\left(\frac{\sigma r}{\lambda^f} + g\right) \tag{34}$$

$$\kappa x_t = \lambda^f Ce^{-\lambda^b t} + \frac{\kappa \lambda^b}{\lambda^f + \lambda^b}\left[e^{-\lambda^f (T_h - t)} - e^{-\lambda^f (T_l - t)}\right]\left(\frac{\sigma r}{\lambda^f} + g\right) \tag{35}$$

$t \geq T_h$:

$$\pi_t = Ce^{-\lambda^b t} + \frac{\kappa}{\lambda^f + \lambda^b}\left[e^{-\lambda^b (t - T_h)} - e^{-\lambda^b (t - T_l)}\right]\left(\frac{\sigma r}{\lambda^b} - g\right) \tag{36}$$

$$\kappa x_t = \lambda^f Ce^{-\lambda^b t} + \frac{\kappa \lambda^f}{\lambda^f + \lambda^b}\left[e^{-\lambda^b (t - T_h)} - e^{-\lambda^b (t - T_l)}\right]\left(\frac{\sigma r}{\lambda^b} - g\right) \tag{37}$$

(38)

$T_l \le t \le T_h$:

$$\pi_t = Ce^{-\lambda^b t} + \frac{\kappa}{\lambda^f + \lambda^b} \times \tag{39}$$
$$\left[\left(\frac{1 - e^{-\lambda^b (t - T_l)}}{\lambda^b} + \frac{1 - e^{-\lambda^f (T_h - t)}}{\lambda^f} \right) \sigma r + \left(e^{-\lambda^b (t - T_l)} - e^{-\lambda^f (T_h - t)} \right) g \right]$$

$$\kappa x_t = -\kappa g + \lambda^f Ce^{-\lambda^b t} + \frac{\kappa}{\lambda^f + \lambda^b} \times \tag{40}$$
$$\left[\left(\frac{\lambda^f}{\lambda^b} (1 - e^{-\lambda^b (t - T_l)}) - \frac{\lambda^b}{\lambda^f} (1 - e^{-\lambda^f (T_h - t)}) \right) \sigma r + \left(\lambda^f e^{-\lambda^b (t - T_l)} + \lambda^b e^{-\lambda^f (T_h - t)} \right) g \right].$$

Figures 1 through 3 plot the case $T_l = 0$, $T_h = T$, and $g = 0$.

To select equilibria with $\pi_0 = 0$ or by $\pi_T = 0$, we solve for the corresponding C, giving

$$\pi_0 = 0: \ Ce^{-\lambda^b t} = -\frac{\kappa}{\lambda^f + \lambda^b} e^{-\lambda^b t} \left(1 - e^{-\lambda^f T} \right) \left(\frac{\sigma r}{\lambda^f} + g \right) \tag{41}$$

$$\pi_T = 0: \ Ce^{-\lambda^b t} = \frac{\kappa}{\lambda^f + \lambda^b} e^{-\lambda^b t} \left(1 - e^{\lambda^b T} \right) \left(\frac{\sigma r}{\lambda^b} - g \right). \tag{42}$$

To plot equilibria, I use these values in (34)-(39).

7.3. Formulas for multipliers

To find the multipliers, I take the derivative with respect to g of the formulas for x_t, (35)-(40), and derivatives of C with respect to g from (41) and (42), evaluated at $g = 0$.

Defining x_{2t} by

$$\kappa x_t = \lambda^f Ce^{-\lambda^b t} + \kappa x_{2t}, \tag{43}$$

we have

$$\left. \frac{\partial x_t}{\partial g} \right|_{g=0} = \frac{\partial}{\partial g} \left(\frac{\lambda^f Ce^{-\lambda^b t}}{\kappa} \right) \Bigg|_{g=0} + \left. \frac{\partial x_{2t}}{\partial g} \right|_{g=0}. \tag{44}$$

The parts are

$$\pi_0 = 0: \ \frac{\partial}{\partial g} \left(\frac{\lambda^f Ce^{-\lambda^b t}}{\kappa} \right) \Bigg|_{g=0} = -\frac{\lambda^f}{\lambda^f + \lambda^b} e^{-\lambda^b t} \left(1 - e^{-\lambda^f T} \right) \tag{45}$$

$$\pi_T = 0: \ \frac{\partial}{\partial g} \left(\frac{\lambda^f Ce^{-\lambda^b t}}{\kappa} \right) \Bigg|_{g=0} = -\frac{\lambda^f}{\lambda^f + \lambda^b} e^{-\lambda^b t} \left(1 - e^{\lambda^b T} \right) \tag{46}$$

and

$$t \leq 0: \left.\frac{\partial x_{2t}}{\partial g}\right|_{g=0} = \frac{\lambda^b}{\lambda^f + \lambda^b}\left(e^{-\lambda^f (T-t)} - e^{\lambda^f t}\right) \tag{47}$$

$$t \geq T: \left.\frac{\partial x_{2t}}{\partial g}\right|_{g=0} = -\frac{\lambda^f}{\lambda^f + \lambda^b}\left(e^{-\lambda^b (t-T)} - e^{-\lambda^b t}\right) \tag{48}$$

$$0 \leq t \leq T: \left.\frac{\partial x_{2t}}{\partial g}\right|_{g=0} = -1 + \frac{1}{\lambda^f + \lambda^b}\left(\lambda^f e^{-\lambda^b t} + \lambda^b e^{-\lambda^f (T-t)}\right) \tag{49}$$

Equation (46) holds the key to large multipliers. The term $e^{\lambda^b T}$ is the only exponent of a positive number in these formulas. As T grows, this term grows without bound.

7.4. Formulas for forward guidance

The postponed interest rate rise solution comes from adding up two cases of (34)-(40), $T_l = 0$, $T_h = T$ with $z_1 = \kappa\sigma(i - r) = 2\%$ and $T_l = T$, $T_h = T + \tau$ using $z_2 = -2\%$. We obtain:

$$\pi_t = Ce^{-\lambda^b t} + \frac{w_t}{\lambda^f + \lambda^b} \tag{50}$$

$$\kappa x_t = -\kappa g_t + \lambda^f C e^{-\lambda^b t} + \frac{v_t}{\lambda^f + \lambda^b} \tag{51}$$

where

$$t < 0: w_t = \frac{z_1}{\lambda^f}\left(1 - e^{-\lambda^f T}\right)e^{\lambda^f t} + \frac{z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right)e^{\lambda^f (t-T)} \tag{52}$$

$$0 < t < T: w_t = \frac{z_1}{\lambda^b}\left(1 - e^{-\lambda^b t}\right) + \frac{z_1}{\lambda^f}\left(1 - e^{-\lambda^f (T-t)}\right) + \frac{z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right)e^{\lambda^f (t-T)} \tag{53}$$

$$T < t < T+\tau: w_t = \frac{z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right)e^{-\lambda^b t} + \frac{z_2}{\lambda^b}\left(1 - e^{-\lambda^b (t-T)}\right) + \frac{z_2}{\lambda^f}\left(1 - e^{-\lambda^f (T+\tau-t)}\right) \tag{54}$$

$$t > T+\tau: w_t = \frac{z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right)e^{-\lambda^b t} + \frac{z_2}{\lambda^b}\left(e^{\lambda^b \tau} - 1\right)e^{-\lambda^b (t-T)} \tag{55}$$

$$t < 0 : v_t = -\frac{\lambda^b z_1}{\lambda^f}\left(1 - e^{-\lambda^f T}\right) e^{\lambda^f t} - \frac{\lambda^b z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right) e^{\lambda^f (t-T)} \tag{56}$$

$$0 < t < T : v_t = \frac{\lambda^f z_1}{\lambda^b}\left(1 - e^{-\lambda^b t}\right) - \frac{\lambda^b z_1}{\lambda^f}\left(1 - e^{-\lambda^f (T-t)}\right) - \frac{\lambda^b z_2}{\lambda^f}\left(1 - e^{-\lambda^f \tau}\right) e^{\lambda^f (t-T)} \tag{57}$$

$$T < t < T+\tau : v_t = \frac{\lambda^f z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right) e^{-\lambda^b t} + \frac{\lambda^f z_2}{\lambda^b}\left(1 - e^{-\lambda^b (t-T)}\right) - \frac{\lambda^b z_2}{\lambda^f}\left(1 - e^{-\lambda^f (T+\tau-t)}\right) \tag{58}$$

$$t > T+\tau : v_t = \frac{\lambda^f z_1}{\lambda^b}\left(e^{\lambda^b T} - 1\right) e^{-\lambda^b t} + \frac{\lambda^f z_2}{\lambda^b}\left(e^{\lambda^b \tau} - 1\right) e^{-\lambda^b (t-T)} \tag{59}$$

I then pick $C = 0$, the C that delivers $\pi_{T+\tau} = 0$ and the C that delivers $\pi_0 = 0$.